NYU NYO-9351 c.1.

LUDWIG

The singularities of the Riemann function.

AEC Computing and Applied Mathematics Center
Institute of Mathematical Sciences
New York University

TID-4500
15th Ed.

NYO-9351
PHYSICS

THE SINGULARITIES OF THE RIEMANN FUNCTION

by

Donald Ludwig

January 1, 1961

Contract No. AT(30-1)-1480

NYO-9351
PHYSICS AND
MATHEMATICS

ABSTRACT

This paper deals with the Riemann function for linear hyperbolic systems of first-order equations. The leading term in the singularity of the Riemann function is determined and interpreted. In addition to equations with distinct characteristics, certain equations with multiple characteristics are treated.

TABLE OF CONTENTS

NYO-9351

THE SINGULARITIES OF THE RIEMANN FUNCTION

1. Introduction

We shall deal with the Cauchy problem for linear hyperbolic systems of the first order, with $n+1$ independent variables. Our central problem is the description of the singularities of the Riemann function. In addition to its own intrinsic interest, this problem provides a key to the mathematical theory of wave propagation, since any solution of the Cauchy problem can be represented in terms of the Riemann function. In a sense, the singularities of the Riemann function determine the structure of the dependence of a solution on its initial data.

The analysis of the singularities of the Riemann function also represents a step towards an extension of the method of Hadamard to general linear hyperbolic equations. This method would determine the Riemann function by substitution of a function of the proper form into the differential equation.

Our analysis also provides an approach to problems involving propagation of waves in media where the characteristics have variable multiplicity, including questions of existence and uniqueness. Such problems are frequently ignored in the mathematical literature, although they have great physical interest.

Our principal methods and concepts are drawn from geometrical optics and asymptotic analysis. We shall see that the singularity of the Riemann function in the neighborhood of a point is determined by the local geometry of the ray conoid and the normal surface. At multiple points of the normal surface, the behavior of the vectors annihilated by the characteristic matrix must also be considered.

In dealing with problems involving multiple characteristics, we generally restrict ourselves to equations with constant coefficients and characteristics with at most double multiplicity.1/ Our methods seem to have an immediate extension to more general problems, but the complexity of the method increases with the complexity of the geometry of the normal surface. This is in contrast to existence and uniqueness theorems based on energy inequalities, e.g. for symmetric hyperbolic equations. However, the same difficulties seem to appear when more detailed information about solutions is sought.

We describe only the most singular part of the Riemann function at each point, although the other singularities can be calculated from our formulas.2/

First we describe our results for strongly hyperbolic equations with constant coefficients, whose characteristics

---

1/ We also must exclude conical points of the normal surface, or points where the characteristic nullvectors do not vary smoothly.

2/ Similar formulas have been obtained by J. Leray [11], [12], [13].

have uniform multiplicity. Let the positive direction of the normal to the ray cone point towards the origin. Then at regular points of a sheet of the ray cone, the type of singularity of the Riemann function depends upon the sign of the Gauss curvature of the intersection of the sheet with a plane $t = \text{const}$. Call the intersection S.

If the curvature of S is positive, then the singularity of the Riemann function R is analogous to the singularity for the wave equation, i.e.

$$R \sim a\, \delta^{(\frac{n-1}{2})}(d) \quad .$$

Here d is a measure of the distance to S. The factor a will be described presently. If n is odd, then the singularity has support on S itself. If n is even, then the singularity has support on the positive side of S.

If the curvature of S is negative, then for n even

$$R \sim a\, \delta^{(\frac{n-1}{2})}(-d) \quad .$$

Then the singularity has support on the negative side of S. If the curvature of S is negative and n is odd, then

$$R \sim a \log^{(\frac{n+1}{2})} |d| \quad ,$$

i.e. R has a logarithmic singularity on S.

The factor a is a right nullvector of the characteristic matrix associated with S. The magnitude of a is proportional to the square root of the Gauss curvature of S; the factor of proportionality can be determined solely

from knowledge of the tangent plane to S.

At singular points of S, such as cusps, we are able to give complete results only if $n = 2$. For larger values of n, our methods cover many interesting cases, and we conjecture that the following description is correct in all cases. Let $\rho_1, \ldots, \rho_{n-1}$ be the radii of principal curvature of S. Assume that $\rho_1, \ldots, \rho_k$ vanish at the point under consideration, to orders $\alpha_1, \ldots, \alpha_k$ respectively. We permit any of the α_i to be infinite. Let

$$\beta = \sum_{i=1}^{k} \left(\frac{1}{2} - \frac{1}{\alpha_i + 2}\right) .$$

Then

$$R \sim a\, \delta^{\left(\frac{n-1}{2} + \beta\right)}(d) ,$$

in the neighborhood of the singular point of S.

If our equation is weakly hyperbolic, i.e. if a double root of the characteristic equation corresponds to only one linearly independent right nullvector of the characteristic matrix, then R has one less derivative on the corresponding sheets of the ray cone.

For equations with variable coefficients, slight modifications in our descriptions are required. First we consider "planelike" characteristic surfaces P. These are characteristics whose intersection with the hyperplane $t = 0$ is an $(n-1)$-dimensional plane passing through the origin. The set of all such P envelopes the ray conoid with vertex

at the origin. As before, let S be the intersection of a sheet of the ray conoid with a hyperplane $t = \text{const.}$ At each point of S there is a planelike surface P which is tangent to S. Instead of the principal curvatures of S, we consider the principal relative curvatures of S and P. With this modification, all of the results described above apply to equations with variable coefficients. We also find that the factor a is proportional to the square root of the local ray density on S; the factor of proportionality depends only upon the bicharacteristic strip tangent to S.

Finally, we consider strongly hyperbolic equations with constant coefficients, whose normal surfaces have double points. In general, a double point of the normal surface is mapped onto two rays. We find that the Riemann function is, in general, singular on the convex hull of these rays, and that the singularity is given by

$$R \sim a\, \delta^{(\frac{n-2}{2})}(d) .$$

Examples to illustrate our results are given in Section 2. For characteristics with uniform multiplicity, the construction of the singular part of R is given in Sections 3 and 4. The method of stationary phase is explained in Section 5, and the interpretation of results is given in Section 6. Equations whose characteristics have non-uniform multiplicity are discussed in Sections 7 and 8.

This work has been stimulated by the forthcoming Courant-Hilbert, Methods of Mathematical Physics, II. I have benefited from many discussions with R. Courant, J. B. Keller, P. D. Lax and P. Ungar.

2. Examples

In this section, we shall illustrate our results by means of examples. For simplicity, we restrict ourselves to equations with constant coefficients.

First we shall review some definitions and ideas. Consider a linear first-order system of k equations with k unknowns:

$$\mathcal{L} u \equiv \sum_{\nu=0}^{n} A^{\nu} \frac{\partial u}{\partial x^{\nu}} + Bu = 0 .$$

Here u is an unknown vector function of the $n+1$ variables $x^0, x^1, \ldots, x^n$, $(x^0 = t)$ and A^{ν} and B are square matrices. We assume that $A^0 = I$, the identity matrix.

The characteristic equation is a restriction on the direction of the normal of a surface element; it therefore refers to the dual space with coordinates $(\lambda, \xi_1, \ldots, \xi_n)$. The characteristic equation is

$$\det \left(\lambda A^0 + \sum_{\nu=1}^{n} \xi_{\nu} A^{\nu}\right) = 0 .$$

The matrix $\lambda A^0 + \sum_{\nu=1}^{n} \xi_{\nu} A^{\nu}$ is called the characteristic matrix. The vectors which it annihilates are called characteristic nullvectors. The locus of points satisfying

the characteristic equation forms a cone in λ, ξ space, called the <u>normal cone</u> of the equation. The intersection of this cone and the hyperplane $\lambda = -1$ is called the <u>normal surface</u>. The <u>ray cone</u> is the envelope of all characteristic planes which pass through the origin, i.e. all planes

$$\lambda t + x \cdot \xi = 0 ,$$

where (λ, ξ) belongs to the normal cone. The <u>ray surface</u> is the intersection of the ray cone and the hyperplane $t = 1$.

We shall be especially interested in the relationship between the normal surface and the ray surface. The simplest and most usual situation occurs in the neighborhood of an isolated sheet of the normal surface, or where two sheets coincide completely. In this case, points of the normal surface are mapped onto points of the ray surface by means of the following construction: Choose a point P on the normal surface, and draw the tangent plane Π at P. Let Q be the foot of the normal to Π drawn from the origin, and let S be the reflection of Q in the unit sphere. S is the point on the ray surface associated with P. The shape of the ray surface at S is determined by the principal curvature vectors of the normal surface at P. These curvature vectors also determine the singularity of the Riemann function at S.

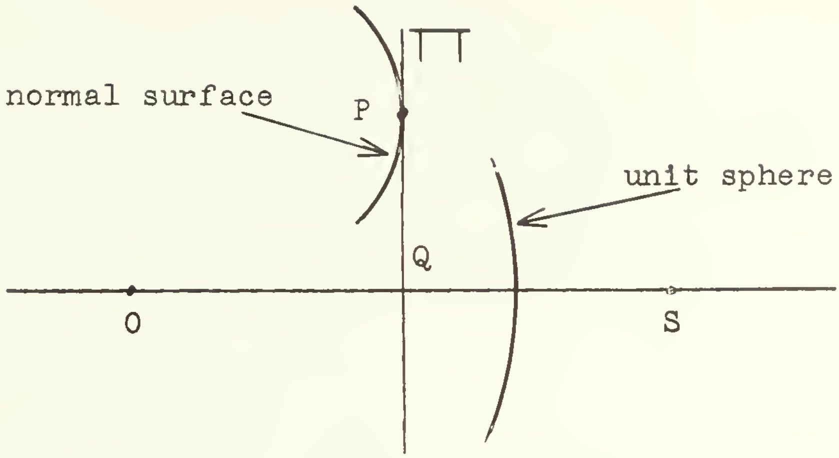

Figure 1

If P is a point where the normal surface intersects itself, then the preceding construction maps P onto two or more points of the ray surface. If the normal surface is shaped like a cone with vertex at P, then the construction maps P onto infinitely many points of the ray surface. In general, if a single point of the normal surface is mapped onto a set of points of the ray surface, then the Riemann function is singular on the convex hull of that point set.

A. The wave equation

Consider the wave operator

$$\Box = \frac{\partial^2}{\partial t^2} - \sum_{\nu=1}^{n} \frac{\partial^2}{(\partial x^\nu)^2} .$$

The corresponding Riemann function $R(t,x,t_0,x_0)$ satisfies the conditions:

$$\Box R = \delta(t-t_0,x-x_0) = \delta(t-t_0)\,\delta(x-x_0) ,$$

$$R = 0 \qquad (t < t_0) .$$

Here $\delta(t-t_0,x-x_0)$, $\delta(t-t_0)$ and $\delta(x-x_0)$ represent the Dirac distribution in $n+1$, 1 and n dimensions respectively. Since the wave operator has constant coefficients, it is sufficient to set $t_0 = 0$, $x_0 = 0$.

We recognize that R is the solution of a radiation problem: the solution is given explicitly in [2], p. 408, by setting $g(\quad) = \delta(\quad)$. We shall also obtain the solution in Section 4. Let $r = |x|$. Then

$$R(t,x) = \frac{(-1)^{\frac{n-3}{2}}}{4\pi^{\frac{n-1}{2}}} \left(\frac{d}{dr^2}\right)^{\frac{n-3}{2}} \frac{\delta(t-r)}{r} \qquad (n \text{ odd})$$

$$R(t,x) = \frac{(-1)^{\frac{n-2}{2}}}{2\pi^{n/2}} \left(\frac{d}{dr^2}\right)^{\frac{n-2}{2}} \int_r^t \frac{\delta(t-\tau)\,d\tau}{\sqrt{\tau^2-r^2}} \qquad (n \text{ even}).$$

Using the properties of the delta function, we may combine these formulas for even and odd n:

$$(2.1) \qquad R(t,x) = \begin{cases} \dfrac{1}{2\pi^{\frac{n-1}{2}}}\, \delta^{(\frac{n-3}{2})}(t^2-r^2) & (t > 0) \\ 0 & (t < 0) \end{cases}$$

If n is even, fractional derivatives of the delta function are involved. By definition,

$$\delta^{(-\frac{1}{2})}(s) = \frac{1}{\sqrt{\pi}} \int_{\infty}^{s} \frac{\delta(\sigma)\,d\sigma}{\sqrt{s-\sigma}},$$

$$= \begin{cases} 0 & (s < 0) \\ \dfrac{1}{\sqrt{\pi}}\dfrac{1}{\sqrt{s}} & (s > 0) \end{cases}$$

Thus we can immediately recognize the difference between odd and even dimensions; if n is odd and $n \geq 3$, then R is different from zero only on the ray cone: $t^2 = r^2$. This is another way of stating that the wave equation satisfies Huygens' principle. If n is even, then R is different from zero in the solid half-cone $t \geq r$, since fractional derivatives of the δ-function are different from zero on a half-line.

What we want to especially notice about equation (2.1) is that R behaves like an $\frac{n-3}{2}$-th derivative of the δ-function as we cross the ray cone: $t^2 = r^2$. The results of Hadamard show that the Riemann function corresponding to a general linear hyperbolic equation of second order has the same singularity on the ray conoid. Thus we might be led to conjecture that the Riemann function always has this type of singularity, if appropriate adjustment is made for the order of the equation, i.e.

$R \sim \delta^{(\frac{n-1}{2})}$ for first-order systems

$R \sim \delta^{(\frac{n-3}{2})}$ for second-order equations or systems

$R \sim \delta^{(\frac{n-5}{2})}$ for third-order equations or systems,

etc.

Our next example will show that this is not always the case.

B. <u>The equations of hydromagnetics</u>

Now we consider the linearized equations of hydromagnetics, with constant coefficients. We shall not attempt to construct the Riemann function for these equations,[1] but merely describe the normal surface, the ray surface, and the singularities of the Riemann function. Our procedures and results will be justified in Section 6.

The equations for small hydromagnetic disturbances are as follows:

$$\frac{\partial \rho}{\partial t} + \rho_0 \operatorname{div} u = 0$$

$$\rho_0 \frac{\partial u}{\partial t} + a^2 \operatorname{grad} \rho - \mu \, (\operatorname{curl} H) \times H_0 = 0 ,$$

$$\frac{\partial H}{\partial t} - \operatorname{curl} \, (u \times H_0) = 0 .$$

Here the underlying flow is at rest, with constant density ρ_0, magnetic field vector H_0, and sound speed a.

The characteristic equation for this system is as follows:

$$\lambda \, [\lambda^2 - (A \cdot \xi)^2] \, [\lambda^4 - (a^2 + A^3) \, \lambda^2 \xi^2 + a^2 \xi^2 (A \cdot \xi)^2] = 0 ,$$

where $A = \sqrt{\frac{\mu}{\rho_0}} \, H_0$. The factor λ is eliminated from the

---

[1] These equations are discussed in Friedrichs and Kranzer [5], H. Grad [6], and Bazer and Fleischman [1]. Recently a partial solution for 3-dimensional propagation has been given by F. G. Friedlander [4], and a complete solution for 2-dimensional propagation has been given by H. Weitzner [16].

characteristic equation if we restrict the initial data by the condition that div $H = 0$, and then eliminate one unknown. We consider the case where $A^2 > a^2$.

The normal surface for these equations is represented in Figure 2. It is a surface of revolution, with the axis pointing in the direction of H_0.

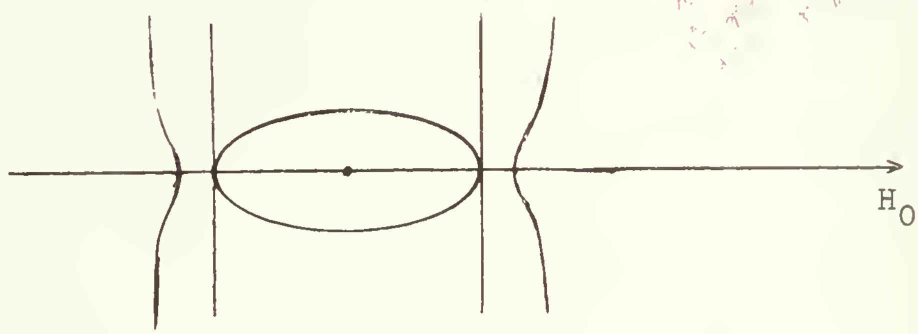

Figure 2

The ray surface is represented in Figure 3. It is also a surface of revolution. The outer shell corresponds

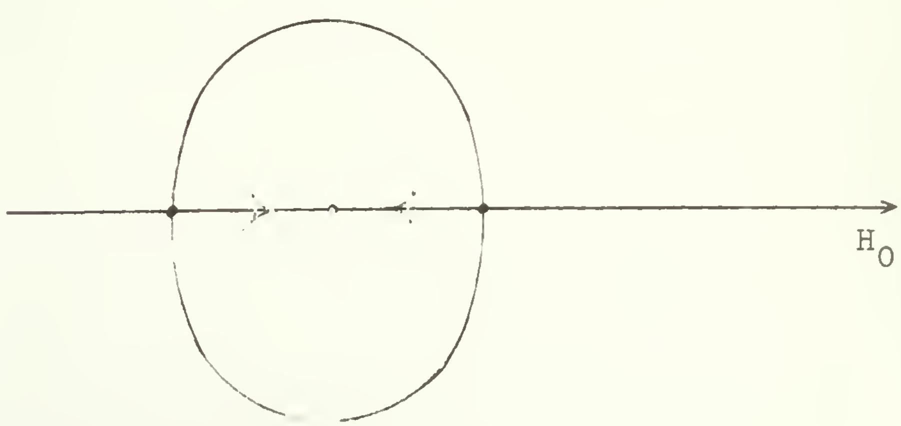

Figure 3

to the inner oval of the normal surface. The parallel planes tangent to the oval are mapped onto two points on the H_0-axis. The two outer sheets of the normal surface are mapped onto the cusped triangles inside the outer shell of the ray surface.

First we consider the case of two-dimensional propagation, i.e. when the surfaces are as shown, and not revolved around the H_0-axis. Since the outer shell of the ray surface is convex, the singularity of the Riemann function on the outer shell is analogous to the singularity for the wave equation, i.e.

$$R \sim \delta^{(1/2)}(d) \sim [\frac{1}{d^{3/2}}] \quad \text{inside the ray surface,}$$

$$R \sim 0 \quad \text{outside the ray surface.}$$

Here d is a measure of the distance to the ray surface. The notation $[\frac{1}{d^{3/2}}]$ indicates that we are dealing with a distribution, i.e. that we must take finite parts of integrals involving R.

At the intersection of the H_0-axis and the outer shell of the ray surface, the Riemann function has a sharper singularity due to the fact that an entire sheet of the normal surface is mapped onto a point. This situation is analogous to a perfect focus in optics. The singularity is raised by a 1/2-derivative:

$$R \sim \delta^{(1)}(d) ,$$

in the neighborhood of these points. Introducing polar coordinates, we recognize that this is a singularity like the delta-function in two dimensions. This is to be expected, since the ray surface corresponding to an ordinary differential equation consists of a single ray.

The behavior of the Riemann function on the cusped triangles is more complicated. The outer cusps correspond to the points of inflection of the outer sheets of the normal surface. Since the radius of curvature has a non-vanishing derivative at the cusp, the singularity is raised by a 1/6-th derivative $(1/2 - 1/3 = 1/6)$. This situation is analogous to an imperfect focus in optics. Thus we have

$$R \sim \delta^{(2/3)}(d) \sim \frac{1}{[d^{5/3}]} ,$$

in the neighborhood of the outer cusps.

On the curved portion of the ray surface between the outer cusps, the curvature vector points away from the origin. Here we must expect a different kind of singularity than on the outer sheet of the ray surface. We find:

$R \sim \delta^{(1/2)}(d)$ on the side away from the origin

$R \sim 0$ on the side toward the origin.

Thus the singularity is on the opposite side of the surface than we would expect from our knowledge of the wave equation.

The inner cusps of the ray surface correspond to the point of the normal surface at infinity. Since the outer sheets lie inside their asymptotes, the ray surface has cusps. Here again we find

$$R \sim \delta^{(2/3)}(d) ,$$

in the neighborhood of the inner cusps. On the remainder of the ray surface we have

$R \sim \delta^{(1/2)}$ on the side toward the origin

$R \sim 0$ on the side away from the origin,

since the corresponding curvature vector of the ray surface points inwards.

We may summarize our results in the following diagram:

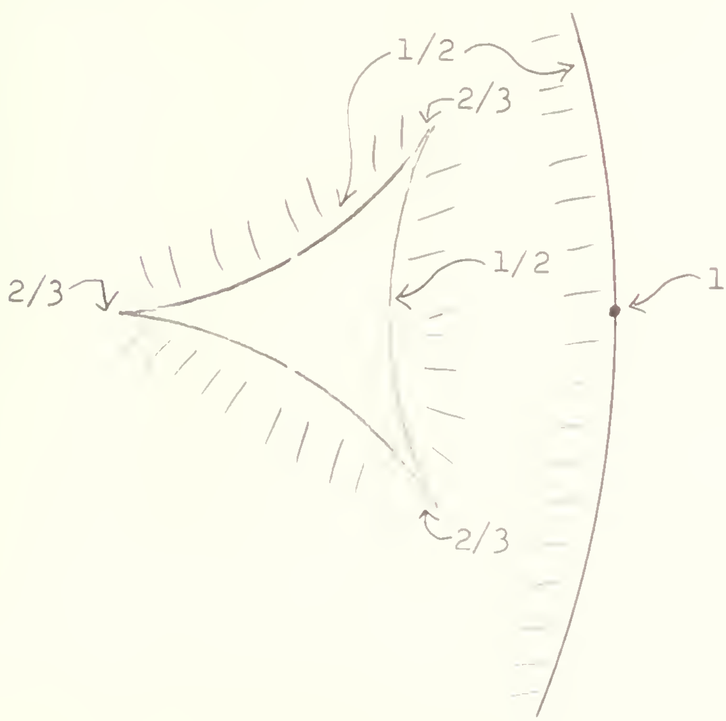

Figure 4

Note that the Riemann function is not singular inside the cusped regions. Weitzner's solution shows that in fact the Riemann function vanishes inside the cusped regions.

The singularities in the three-dimensional case may be obtained by analogous considerations. We must bear in mind, however, that the ray surface has an additional curvature vector pointing toward the H_0-axis. On the outer sheet of the ray surface, the singularity is given by analogy with the wave equation:

$$R \sim \delta^{(1)}(d) \quad \text{on the outer sheet.}$$

However, on the H_0-axis, there is an additional point singularity due to the plane sheets of the normal surface. At these points, the singularity is raised by one derivative $(1 = 2 \cdot \frac{1}{2})$:

$$R \sim \delta^{(2)}(d) \quad ,$$

in the neighborhood of these points. Again introducing polar coordinates, we recognize the delta-function in three dimensions.

The outer circular cusps of the ray surface again correspond to the points of inflection of the cross-section of the normal surface. Here again the singularity is enhanced by a 1/6-th derivative:

$$R \sim \delta^{(7/6)}(d) \quad \text{near the cusps.}$$

On the portion of the ray surface bounded by the

circular cusps, both curvature vectors point away from the origin. Thus here again we have

$$R \sim \delta^{(1)}(d) \quad ,$$

in analogy with the wave equation.

The inner cusps of the ray surface correspond to an imperfect focus in three dimensions. Accordingly, the singularity is enhanced by a 1/3-derivative $(1/3 = 2(\frac{1}{2} - \frac{1}{3}))$:

$$R \sim \delta^{(4/3)}(d) \quad \text{near the inner cusps.}$$

On the remainder of the ray surface, one curvature vector points inwards, and the other outwards. Accordingly we cannot be guided by the wave equation: we find that

$$R \sim \log^{(2)}(d) \sim [\frac{1}{d^2}] \quad .$$

This means that we must take finite parts of integrals involving $1/d^2$. If we were to take logarithmic parts of these integrals, we would have a singularity like $\delta^{(1)}(d)$. Thus, although these distributions have many formal similarities, they must be carefully distinguished. In particular, $\log^{(2)}(d)$ does not have support only at $d = 0$; the Riemann function is singular inside the cusped regions as well as outside.

We may summarize our results in the following schematic diagram:

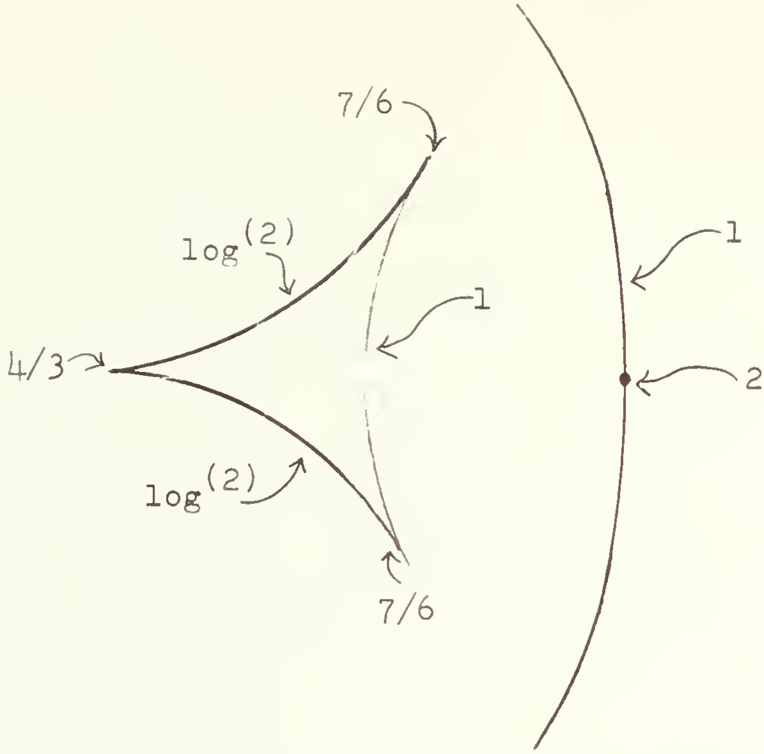

Figure 5

C. Equations with multiple characteristics

The theory of hyperbolic systems with multiple characteristics is still far from complete. We shall confine ourselves to some brief remarks and simple examples, which illustrate the theory to be developed in Sections 7 and 8.

If two or more roots of the characteristic equation coincide, then the corresponding nullvectors of the characteristic matrix may not be linearly independent. In this case the lower order terms in the equation (the matrix B) determine whether or not the Cauchy problem is well posed. For example, the system

$$\begin{aligned} u_t - v_x &= 0 \\ v_t &= 0 \end{aligned} \tag{2.2}$$

is hyperbolic, while the system

$$u_t - v_x = 0$$

$$v_t - u = 0 ,$$

is equivalent to the equation $u_{tt} = u_x$, which is parabolic. In Section 3, we shall obtain a simple condition on B which is necessary and sufficient for the Cauchy problem to be well-posed, for equations whose characteristics have constant multiplicity. If such conditions on B are necessary, then the equation is called weakly hyperbolic, otherwise strongly hyperbolic. M. Yamaguti and K. Kasahara [17] have shown that a system is strongly hyperbolic if and only if the nullvectors of the characteristic matrices for fixed $\xi_1,\ldots,\xi_n$ form a (uniformly) complete set.

The Riemann function for a weakly hyperbolic system is more singular than for a strongly hyperbolic system. On the appropriate sheets of the ray surface, the singularity of the Riemann function is enhanced by a number of derivatives equal to the defect in the number of nullvectors of the characteristic matrix. Thus the Riemann function for the system (2.2) is given by

$$R = \begin{pmatrix} \delta(x) & t\,\delta^1(x) \\ 0 & \delta(x) \end{pmatrix} .$$

In general, if the normal surface has double points, then the Riemann function is singular not only on the ray

surface, but also on certain manifolds which are supported but not envelopped by the corresponding characteristic surfaces.[1] Thus the Riemann function is generally singular on the convex hull of each sheet of the ray surface. If the normal surface has a double point, then the singularity on the corresponding part of the hull differs from what we might expect from consideration of the wave equation. If the equation is strongly hyperbolic, then the singularity is weaker by 1/2-derivative; if the equation is weakly hyperbolic, then the singularity is stronger by 1/2-derivative.

For example, consider the strongly hyperbolic system

$$u_t - u_x - v = 0 ,$$

$$v_t - v_y = 0 ,$$

with Cauchy data

$$u(0,x,y) = f(x,y)$$

$$v(0,x,y) = g(x,y) .$$

The solution is

$$u(t,x,y) = f(t+x,y) + \int_0^t g(x+t-s,s+y)\, ds ,$$

$$v(t,x,y) = g(x,t+y) .$$

Thus the Riemann function is given by

$$R = \begin{pmatrix} \delta(t+x)\ \delta(y) & H(-x)H(-y)\ \delta(x+y+t) \\ 0 & \delta(x)\ \delta(y+t) \end{pmatrix} ,$$

[1] This phenomenon is connected with the fact that singularities in the solutions propagate between, as well as along, rays.

where $H(x)$ is the Heaviside function ($H(x) = 1$ for $x > 0$, $H(x) = 0$ for $x < 0$). The ray surface is shown in Figure 6. R has a singularity like $\delta(d)$ across the segment of the line $x+y+1 = 0$ cut off by the x and y axes. At the endpoints of this segment R behaves like the delta-function of two variables, corresponding to the fact that the ray cone consists of just two rays.

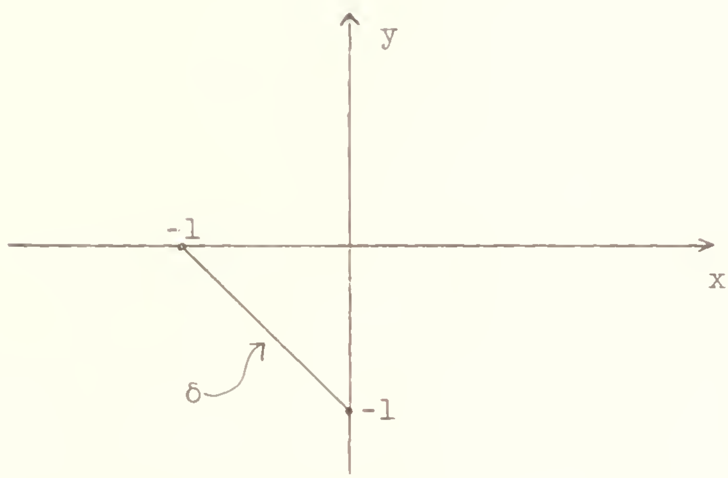

Figure 6

As an example of a weakly hyperbolic system, we consider

$$u_t + u_y - v_x = 0$$

$$v_t - v_y = 0 ,$$

with Cauchy data

$$u(0,x,y) = f(x,y)$$

$$v(0,x,y) = g(x,y) .$$

The solution is

$$u = f(x,y-t) + \int_0^t g_x(x,y-t+2s)\,ds\ ,$$

$$v = g(x,t+y)\ ,$$

and the Riemann function is given by

$$R = \begin{pmatrix} \delta(x)\ \delta(y-t) & \delta^1(x)\ H(t^2-y^2) \\ 0 & \delta(x)\ \delta(t+y) \end{pmatrix}$$

The ray surface is shown in Figure 7. Across the segment $x = 0$, $y^2 < 1$, R has a singularity like $\delta^{(1)}(d)$, and at the endpoints, R behaves like the delta-function of two variables.

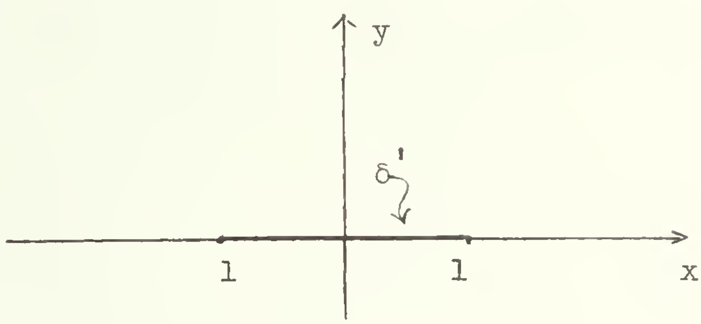

Figure 7

3. Propagation of Singularities along Characteristics with Uniform Multiplicity

In this section we shall consider the preliminary problem of calculating the propagation of singularities along regular characteristic surfaces, where the normal cone consists of isolated sheets, or sheets which coincide completely. This problem has been treated in D. Ludwig [14],

for strongly hyperbolic equations. We shall first give a brief summary of the results of [14], and a geometrical interpretation of the leading term in the expansion of the solution. Next we shall treat a special class of weakly hyperbolic equations with constant coefficients. In Section 4, we shall show how the expansions derived in this section can be used to construct the Riemann function.

A. Strongly hyperbolic systems.

We consider a strongly hyperbolic system with variable coefficients:

$$\mathcal{L}u \equiv \sum_{\nu=0}^{n} A^{\nu}(x) \frac{\partial u}{\partial x^{\nu}} + B(x)\, u = 0 . \tag{3.1}$$

Here $u(x)$ is a vector with k components, and A^{ν} $(\nu = 0,1,\ldots,n)$ and B are $k \times k$ matrices which are C^{∞} functions of x. We assume that $A^0 = I$. We shall construct solutions of (3.1) of the form

$$u = \sum_{j=0}^{\infty} f_j(\phi(x))\, a^j(x) . \tag{3.2}$$

More precisely, we wish to determine the function $\phi(x)$ and vectors $a^j(x)$ $(j = 0,1,\ldots)$ in such a way that if

$$\frac{d}{d\phi} f_j(\phi) = f_{j-1}(\phi) \qquad (j = 1,2,\ldots) , \tag{3.3}$$

then (3.2) is a formal solution of (3.1) for arbitrary choice of $f_0(\phi)$. Then we may choose $f_0(\phi)$ to be a distribution, and (3.2) will be a (formal) weak solution

of (3.1). Since $f_j(\phi)$ becomes as smooth as we like for sufficiently large j, we may break off our series after a finite number of terms and reduce a problem involving singular data to a problem with arbitrarily smooth data.

Using the relations (3.3), we substitute (3.2) into the differential equation and collect terms:

$$\mathcal{L}u = f_0'(\phi)[A^\nu\phi_\nu a^0] + \sum_{j=0}^{\infty} f_j(\phi)[A^\nu\phi_\nu a^{j+1} + \mathcal{L}a^j] = 0 .$$

Here and in the sequel, we shall always sum over the index ν from 0 to n. Our equation will be formally satisfied for arbitrary $f_0(\phi)$ if we set each term inside the brackets equal to zero:

$$A^\nu \phi_\nu a^0 = 0 , \tag{3.4}$$

$$A^\nu \phi_\nu a^{j+1} + \mathcal{L}a^j = 0 \qquad j = 0,1,\ldots . \tag{3.5}$$

From (3.4), we conclude that

$$\det (A^\nu \phi_\nu) = 0 . \tag{3.6}$$

Thus the surfaces $\phi = \text{const.}$ must be characteristic. Let the rank of $A^\nu\phi_\nu$ be $k-r$. Then there are r linearly independent right and left nullvectors of $A^\nu\phi_\nu$:

$$A^\nu \phi_\nu R^\beta = 0 \qquad \beta = 1,\ldots,r ,$$

$$L^\gamma A^\nu \phi_\nu = 0 \qquad \gamma = 1,\ldots,r .$$

We may choose L^γ $(\gamma = 1,\ldots,r)$ such that

$$L^\gamma \cdot R^\beta = \delta^{\gamma\beta} \qquad \alpha,\beta = 1,\ldots,r \; .$$

We also see from (3.4) that a^0 must be a linear combination of $R^1,\ldots,R^r$:

$$a^0 = \sum_{\beta=1}^{r} \sigma^\beta R^\beta \; .$$

Setting $j = 0$ in (3.5) and multiplying by L^γ $(\gamma = 1,\ldots,r)$, we obtain r equations tc be satisfied by $\sigma^1,\ldots,\sigma^r$:

$$\text{(3.7)} \qquad L^\gamma \sum_{\beta=1}^{r} \mathcal{L}(\sigma^\beta R^\beta) = 0 \qquad \gamma = 1,\ldots,r \; .$$

We shall see presently (see 3.20) that this is a system of first order partial differential equations with proportional principal parts. Hence (3.7) can be solved by the method of characteristics.

Once σ^β has been determined, we may obtain a^1 modulo a right nullvector of $A^\nu \phi_\nu$ from (3.5). This nullvector can be determined from a system of inhomogeneous equations analogous to (3.7). The other coefficients $a^2, a^3, \ldots,$ can be obtained in a similar fashion.

Now we can attack the Cauchy problem for strongly hyperbolic equations. We impose Cauchy data for (3.1) in the form:

$$\text{(3.8)} \quad u(0,x^1,\ldots,x^n) = \sum_{j=0}^{\infty} f_j(\phi(0,x^1,\ldots,x^n))\, b^j(x^1,\ldots,x^n) \; .$$

This Cauchy problem cannot in general be solved by means of a single expansion (3.2) since, for example, b^0

may not be a nullvector of a characteristic matrix. First we construct all solutions of the characteristic equation with initial data

$$\phi^{\alpha}(0,x^1,\ldots,x^n) = \phi(x^1,\ldots,x^n) .$$

If $\mathcal{L}$ is hyperbolic and $x^0 = t = 0$ is spacelike, then $\phi^{\alpha}(x)$ are all real if $\phi(x^1,\ldots,x^n)$ is real. If $\mathcal{L}$ is strongly hyperbolic, then the associated right nullvectors form a complete set, i.e. any vector with k components can be represented as a linear combination of these null-vectors. Thus we write

$$u = \sum_{\alpha=1}^{m} \sum_{j=0}^{\infty} f_j(\phi^{\alpha}(x))\, a^{\alpha j}(x) . \tag{3.9}$$

From the initial condition (3.8), we see that

$$\sum_{\alpha=1}^{m} a^{\alpha j}(0,x^1,\ldots,x^n) = b^j(x^1,\ldots,x^n) .$$

This is a system of equations for the k scalars $\sigma^{\alpha\beta j}$ $(\beta = 1,\ldots,r_{\alpha}\, ;\ \alpha = 1,\ldots,m)$, for each value of j. Since the associated nullvectors $R^{\alpha\beta}$ $(\beta = 1,\ldots,r_{\alpha}\, ;\ \alpha = 1,\ldots,m)$ form a complete set, $\sigma^{\alpha\beta j}$ can be determined uniquely on the initial manifold.

B. Interpretation of the leading term in the expansion

By examining the system (3.7), we may give the scalars σ^{β} a geometrical interpretation. If we think of the square of σ^{β} as a measure of energy, then we may state: the

energy of the disturbance is transported along rays.[1] More precisely, σ^β is proportional to the square root of the local density of rays; the factor of proportionality can be determined from knowledge of the first derivatives of the phase function alone. For equations with constant coefficients, the local density of rays is given by the Gauss curvature of the intersection of the phase surface and the plane $t = \text{const}$. This interpretation will be applied to the Riemann function in Section 6.

Our argument is as follows: first we derive the ordinary differential equation satisfied by the ray density. Then a series of calculations enables us to write the system (3.7) in the form

$$(3.10) \quad \frac{d}{ds}(\sigma^\gamma \sqrt{\triangle}) + \sum_{\beta=1}^{r} Q^{\gamma\beta}(\sigma^\beta\sqrt{\triangle}) = 0 \qquad \gamma = 1,\ldots,r \, .$$

Here s is a parameter along the ray, $\triangle^{-1}$ is the local ray density, and $Q^{\gamma\beta}$ depends only upon the first derivatives of ϕ, i.e. $Q^{\gamma\beta}$ is a function defined on the characteristic strip. We shall adopt the summation convention that a repeated index ν is summed from 0 to n, and a repeated index i, j or k is summed from 1 to n.

In order to define the ray density, we consider the canonical system of ordinary differential equations which

---

[1] This is a consequence of the fact that characteristics have uniform multiplicity. The solutions given in Section 7 do not have this property. Another such example is given in D. Ludwig [15].

define the rays. In the neighborhood of a solution of the characteristic equation with uniform multiplicity, we may determine ϕ_t explicitly as a function of $\phi_{x^1}, \ldots, \phi_{x^n}$:

$$\phi_t = \lambda(t, x, \phi_x) . \tag{3.11}$$

Let $\phi(t,x)$ have the initial values

$$\phi(0,x) = \psi(x) .$$

We introduce the parameters $y^1, \ldots, y^n$ in the initial plane by the equations

$$x^i = X^i(y) \qquad i = 1, \ldots, n .$$

We may choose $y^1, \ldots, y^n$ in such a way that

$$y^n = \psi(x) .$$

Then we define $x^\nu(s,y)$ and $p_\nu(s,y)$ $(\nu = 0, 1, \ldots, n)$ by means of the system of ordinary differential equations (Hamilton's equations)

$$\tag{3.12} \left\{ \begin{array}{lll} \dot{t}(s,y) = 1 & & \\ \dot{x}^i(s,y) = -\dfrac{\partial \lambda(t,x,p)}{\partial p_i} & & i = 1, \ldots, n , \\ \dot{p}_0(s,y) = \dfrac{\partial \lambda(t,x,p)}{\partial t} & & \\ \dot{p}_i(s,y) = \dfrac{\partial \lambda(t,x,p)}{\partial x^i} & & i = 1, \ldots, n , \end{array} \right.$$

with the initial conditions

$$t(0,y) = 0 \,,$$

$$x^i(0,y) = X^i(y) \qquad i = 1,\ldots,n$$

$$p_i(0,y) = \frac{\partial\psi(X(y))}{\partial x^i} \qquad i = 1,\ldots,n$$

$$p_0(0,y) = \lambda(0,X(y),p_i) \,.$$

Then, since y is constant along a ray, the ratio

$$\frac{\Delta_0}{\Delta} = \frac{\frac{\partial(X)}{\partial(y)}}{\frac{\partial(x(s,y))}{\partial(y)}} \,,$$

given the ratio of the ray density at $t = s$ to the ray density at $t = 0$. Provided that $\Delta \neq 0$, we may determine y as a function of x, for fixed t.

In analogy to the procedure to obtain the equation satisfied by the Wronskian of a system of ordinary differential equations, a short calculation using (3.12) shows that

$$\dot{\Delta} = -(\lambda_{p_i x^i} + \lambda_{p_i p_j} \frac{\partial p_j}{\partial x^i})\,\Delta \,. \tag{3.13}$$

Here all quantities are considered as functions of y and s.

Now, thinking of y as a function of x and t, we recognize that

$$\phi(t,x) = y^n(t,x) \,,$$

and the gradients of $y^1,\ldots,y^{n-1}$ are tangential to the surfaces ϕ = const. Thus we may define an "inner ray density" as the (n-1)-dimensional volume spanned by the

gradients of $y^1, \ldots, y^{n-1}$. We have

$$(3.14) \qquad \frac{\partial(y)}{\partial(x)} = \frac{1}{\Delta} = \rho |\phi_x| \ ,$$

where ρ is the inner ray density.

Now we proceed to analyze the system (3.7):

$$(3.7) \quad L^\gamma \sum_{\beta=1}^{r} [A^\nu R^\beta \frac{\partial \sigma^\beta}{\partial x^\nu} + (A^\nu \frac{\partial R^\beta}{\partial x^\nu} + B R^\beta) \sigma^\beta] = 0 \ .$$

The vectors L^γ and R^β satisfy the relations

$$(3.15) \qquad A^\nu p_\nu R^\beta(t,x,p) = 0 \ , \qquad \beta = 1, \ldots, r \ ,$$

$$(3.16) \qquad L^\gamma(t,x,p) A^\nu p_\nu = 0 \ , \qquad \gamma = 1, \ldots, r \ ,$$

$$(3.17) \qquad L^\gamma \cdot R^\beta = \delta^{\gamma\beta} \ .$$

Recalling that $A^0 = I$ and $p_0 = \lambda(t,x,p_i)$, we differentiate (3.15) with respect to p_i and p_j:

$$(3.18) \quad (\lambda_{p_i} + A^i) R^\beta + (\lambda + A^j p_j) R^\beta_{p_i} = 0 \ ,$$

$$(3.19) \quad \lambda_{p_i p_j} R^\beta + (\lambda_{p_i} + A^i) R^\beta_{p_j} + (\lambda_{p_j} + A^j) R^\beta_{p_i} + (\lambda + A^k p_k) R^\beta_{p_i p_j} = 0.$$

Multiplying (3.18) by L^γ, we obtain

$$L^\gamma A^i R^\beta = - \delta^{\gamma\beta} \lambda_{p_i} \qquad i = 1, \ldots, n \ ;$$

hence

$$(3.20) \qquad L^\gamma A^\nu R^\beta = \delta^{\gamma\beta} \dot{x}^\nu \qquad \nu = 0, 1, \ldots, n \ .$$

This shows that the direction of differentiation in (3.7) is bicharacteristic. Multiplying (3.19) by $L^\gamma \phi_{ij}$ and summing

over i and j, we obtain

$$(3.21) \quad \lambda_{p_i p_j} \phi_{ij} \delta^{\gamma\beta} + 2L^{\gamma}(\lambda_{p_i} + A^i) R^{\beta}_{p_j} \phi_{ij} = 0 .$$

This equation enables us to replace derivatives of R^{β} by by second derivatives of λ.

We shall require one additional identity: it follows from (3.11) that

$$(3.22) \quad \phi_{tx^i} = \lambda_{x^i} + \lambda_{p_j} \phi_{ij} \qquad i = 1, \ldots, n .$$

Now we can put the system (3.7) into the form (3.10). We shall denote partial derivatives of R^{β} by means of subscripts and total derivatives by means of the symbol $\partial/\partial x^{\nu}$; thus

$$\frac{\partial R^{\beta}}{\partial x^{\nu}} = R^{\beta}_{x^{\nu}} + R^{\beta}_{p_i} \phi_{i\nu} .$$

It follows that

$$P^{\gamma\beta} \equiv L^{\gamma} A^{\nu} \frac{\partial R^{\beta}}{\partial x^{\nu}} + L^{\gamma} B R^{\beta} = L^{\gamma} R^{\beta}_{p_i} \phi_{ti} + L^{\gamma} A^{j} R^{\beta}_{p_i} \phi_{ij} + L^{\gamma} A^{\nu} R^{\beta}_{x^{\nu}} + L^{\gamma} B R^{\beta} .$$

Using (3.22) and (3.21), we have

$$P^{\gamma\beta} \equiv -\frac{\delta^{\gamma\beta}}{2} \lambda_{p_i p_j} \phi_{ij} + L^{\gamma} R^{\beta}_{p_i} \lambda_{x^i} + L^{\gamma} A^{\nu} R_{x^{\nu}} + L^{\gamma} B R^{\beta} .$$

From (3.13), it follows that

$$P^{\gamma\beta} = \delta^{\gamma\beta} \frac{\mathring{\Delta}}{2\Delta} + \frac{1}{2} \delta^{\gamma\beta} \lambda_{p_i x^i} + L^{\gamma} R^{\beta}_{p_i} \lambda_{x^i} + L^{\gamma} A^{\nu} R^{\beta}_{x^{\nu}} + L^{\gamma} B R^{\beta} ,$$

and hence, using (3.20), (3.7) takes the form

$$(3.23)\quad \sum_{\beta=1}^{r} [\delta^{\gamma\beta}\, \overset{\circ}{\sigma}{}^{\beta} + \frac{\dot{\Delta}}{2\Delta}\, \sigma^{\overset{?\gamma}{\beta}} + Q^{\gamma\beta}\, \sigma^{\beta}] = 0\ , \quad \gamma = 1,\ldots,r,$$

where

$$(3.24)\quad Q^{\gamma\beta} = \frac{1}{2}\, \lambda_{p_i x^i}\, \delta^{\gamma\beta} + L^{\gamma} R^{\beta}_{p_i}\, \lambda_{x^i} + L^{\gamma} A^{\nu} R^{\beta}_{x^{\nu}} + L^{\gamma} B\, R^{\beta}\ .$$

Equation (3.23) is clearly equivalent to (3.10).

C. <u>Weakly hyperbolic systems</u>

Now we consider a weakly hyperbolic system with constant coefficients. We deal with plane characteristic surfaces, and assume that only one characteristic has multiplicity two. We can solve the Cauchy problem for this system if we can exhibit two independent solutions corresponding to the double characteristic, since solutions corresponding to the other characteristics may be treated by the method outlined for strongly hyperbolic systems.

We obtain two independent systems by showing that the scalars σ^{j} satisfy a second-order differential equation, instead of a first-order differential equation. In our special case, the factors σ^{j} can be determined from ordinary differential equations along the rays.

We consider, as before,

$$(3.25)\qquad \mathcal{L}u \equiv \sum_{\nu=0}^{n} A^{\nu} \frac{\partial u}{\partial x^{\nu}} + Bu = 0\ ,$$

where $A^{0} = I$. Let ξ be a fixed vector with components

$\xi_1,\dots,\xi_n$. We can construct the Riemann function provided that we can obtain solutions of (3.25) which depend only on x^0 and $\sum_{i=1}^{n} x^i\xi_i$. Henceforth, we shall always sum the repeated index i from 1 to n. We denote x^0 by t, and $(x^1,\dots,x^n)$ by x. We look for solutions of (3.25) of the form

$$(3.26) \qquad u = \sum_{j=-1}^{\infty} f_j(\phi)\, a^j(t,\xi\cdot x) ,$$

where

$$\phi(t,x) = \lambda t + \xi\cdot x .$$

Note that j is summed from -1 to ∞, instead of from 0 to ∞. We assume that ϕ is a double root of the characteristic equation, i.e. $-\lambda$ is a double eigenvalue of $A^i\xi_i$. We assume in addition that $A^i\xi_i$ has only one eigenvector corresponding to λ ; thus there exist vectors R, S, L and M such that

$$(3.27) \qquad A^i\,\xi_i\,R + \lambda\,R = 0 ,$$

$$(3.28) \qquad A^i\,\xi_i\,S + \lambda\,S = R ,$$

$$(3.29) \qquad L\,A^i\,\xi_i + \lambda\,L = 0 ,$$

$$(3.30) \qquad M\,A^i\,\xi_i + \lambda\,M = L .$$

Substituting (3.26) into (3.25), we obtain

$$(3.31) \qquad u = f_{-2}(\phi)\,[A^\nu\phi_\nu a^{-1}] + f_{-1}(\phi)\,[A^\nu\phi_\nu a^0 + \mathcal{L}a^{-1}] + \sum_{j=0}^{\infty} f_j(\phi)\,[A^\nu\phi_\nu a^{j+1} + \mathcal{L}a^j] = 0 .$$

Hence, as before,

$$a^{-1} = \sigma^{-1} R ,$$

and σ^{-1} must satisfy the equation corresponding to (3.7):

$$(3.32) \qquad L \mathcal{L}(\sigma^{-1} R) = L A^{\nu} R \sigma_{\nu}^{-1} + L B R \sigma^{-1} = 0 .$$

By assumption, σ^{-1} depends only upon t and $x \cdot \xi = y$. Thus (3.32) becomes

$$L R \sigma_t^{-1} + L A^i \xi_i R \sigma_y^{-1} + L B R \sigma^{-1} = 0 .$$

But (3.28) and (3.29) imply that $LR = 0$, and (3.27) then implies that $LA^i \xi_i R = 0$. Thus we have

$$L B R \sigma^{-1} = 0 .$$

Since we are interested in non-trivial solutions, we are led to the condition that

$$(3.33) \qquad L B R = 0 .$$

It can be shown that (3.33) is a necessary and sufficient condition for the Cauchy problem for (3.25) to be well-posed.[1] If (3.33) is satisfied, then we can solve for a^0 in terms of σ^{-1}:

$$(\lambda I + A^i \xi_i) a^0 + \mathcal{L}(\sigma^{-1} R) = 0 ,$$

or

---

[1] This follows from A. Lax [10], and from an unpublished note by A. Lax concerning the connection between a first-order system and the corresponding single equation of higher order.

$$(\lambda I + A^i\xi_i)\ a^0 + (\sigma_t^{-1} - \lambda\sigma_y^{-1})\ R + BR\sigma^{-1} = 0\ .$$

Thus

$$(3.34) \qquad -a^0 = (\sigma_t^{-1} - \lambda\sigma_y^{-1})\ S + P\sigma^{-1} + \sigma^0 R$$

where P satisfies

$$(3.35) \qquad (\lambda + A^i\xi_i)\ P = B\ R\ .$$

Note that $LP = MBR$. Then the compatibility condition

$$L\mathcal{L}(a^0) = 0\ ,$$

leads to the equation

$$(3.36) \quad LS\left(\frac{\partial}{\partial t} - \lambda\frac{\partial}{\partial y}\right)^2 \sigma^{-1} + (MBR+LBS)\left(\frac{\partial}{\partial t} - \lambda\frac{\partial}{\partial y}\right)\sigma^{-1}$$
$$+ LBP\sigma^{-1} = 0\ .$$

Equation (3.36) may be interpreted as a second-order ordinary differential equation for σ^{-1}.

The coefficients a^0, a^1, ... may be determined in a similar fashion from (3.31). Since (3.36) is a second-order equation, we may prescribe σ^j and σ_t^j independently for $t = 0$. Thus we essentially have two solutions corresponding to $\phi = \lambda t + \xi\cdot x$, since u has initial values

$$u(0,x) = f_0(\phi(0,x))\ [\sigma^0 R - \sigma_t^{-1} S] + \sum_{j=1}^{\infty} f_j(\phi(0,x))\ a^j\ ,$$

if we prescribe $\sigma^{-1}(0,x) = 0$.

In particular, solutions exist which have one less derivative than their initial values.

4. Construction of the Riemann Function

In this section we shall show how the Riemann function can be constructed, using a method due to R. Courant and P. D. Lax [3]. The singular part of the Riemann function can be determined by means of the expansions derived in Sections 3 and 7. We shall illustrate the method by constructing the Riemann function for the wave equation.

We consider a linear hyperbolic system of first-order equations:

$$\mathcal{L} u = 0 ,$$

with Cauchy data

$$u(0,x) = g(x) .$$

Here $x^0 = t$, and $x = (x^1, \ldots, x^n)$.

The Riemann function $R(t,x;z)$ satisfies the conditions:

$$\mathcal{L} R = 0 \qquad t > 0 ,$$

$$R(0,x;z) = \delta(x-z) I .$$

Then obviously

$$u(t,x) = \int R(t,x;z)\, g(z)\, d(z) .$$

Thus every solution of the Cauchy problem can be represented in terms of the Riemann function. Uniqueness theorems for hyperbolic equations show that $R(t,x;z)$ has compact support in z-space for fixed t and x.

We can construct the Riemann function if we can

decompose the delta-function into a superposition of plane waves. Such a decomposition is given in F. John [7], p. 11:

$$\delta(z) = -\frac{\Delta^{(n+k)/2}}{k!(2\pi i)^n} \int_{|\omega|=1} (z\cdot\omega)^k \log (z\cdot\omega)\, d\omega,$$

where $\log z\cdot\omega$ denotes the complex-valued function defined on the real axis. We choose $k = 1$ if n is odd, $k = 0$ if n is even, and differentiate under the integral sign. Thus we obtain

$$\delta(z) = -\frac{1}{(2\pi i)^n} \int_{|\omega|=1} \log^{(n)}(z\cdot\omega)\, d\omega ,$$

where $\log^{(n)}$ denotes the n-th derivative of the log function, in the sense of distributions.

Now let $w(t,x;z,\omega)$ satisfy the following conditions:

$$\mathcal{L}w = 0 \qquad\qquad t > 0$$

$$w(0,x;z,\omega) = -\frac{1}{(2\pi i)^n} \log^{(n)}((x-z)\cdot\omega) .$$

Then

$$R(t,x;z) = \int_{|\omega|=1} w(t,x;z,\omega)\, d\omega .$$

Obviously, since the initial date are real, the imaginary part of R vanishes.

The most singular part of R can be obtained from the expansions given in Sections 3 and 8:

$$(4.1) \quad R \sim \sum_{\alpha=1}^{k} \int_{|\omega|=1} \log^{(n)}(\phi^{\alpha}(t,x;z,\omega))\, a^{\alpha}(t,x;z,\omega)\, d\omega.$$

Here we use only the first term in the expansion of w, in the simplest case where the characteristics are distinct. Analogous formulas hold for the other cases. Note that if the operator $\mathcal{L}$ has constant coefficients, $w(t,x;z,\omega)$ will depend only upon t and $(x-z)\cdot\omega$.

The wave equation

Now we illustrate our method by applying it to the wave equation. The Riemann function for the wave equation satisfies the following conditions:

$$\frac{\partial^2}{\partial t^2} R - \sum_{k=1}^{n} \frac{\partial^2 R}{(\partial x^k)^2} = 0 \qquad t > 0$$

$$R(0,x;z) = 0$$

$$\frac{\partial}{\partial t} R(0,x;z) = \delta(x-z) .$$

The corresponding function $w(t,x;z,\omega)$ satisfies

$$\frac{\partial^2}{\partial t^2} w - \sum_{k=1}^{n} \frac{\partial^2 w}{(\partial x^k)^2} = 0 \qquad t > 0$$

$$w(0,x;z,\omega) = 0$$

$$\frac{\partial}{\partial t} w(0,x;z,\omega) = -\frac{1}{(2\pi i)^n} \log^{(n)}((x-z)\cdot\omega) .$$

Since w and R are functions of $x-z$, we may set $z = 0$, with no loss of generality. We shall use the superscript n to indicate that we have n space dimensions. Obviously, since $w^{(n)}$ satisfies the wave equation in two independent variables,

$$w^{(n)}(t,x;\omega) = -\frac{1}{(2\pi i)^n}[\log^{(n-1)}(t+x\cdot\omega)-\log^{(n-1)}(-t+x\cdot\omega)].$$

Hence,

$$(4.2)\quad R^{(n)}(t,x) = -\frac{1}{2(2\pi i)^n}\int_{|\omega|=1}[\log^{(n-1)}(t+x\cdot\omega)-\log^{(n-1)}(-t+x\cdot\omega)]d\omega.$$

This is an integral of plane waves over the unit sphere.

By a well-known formula [7], p. 8,

$$(4.3)\quad R^{(n)}(t,x) = -\frac{\Omega_{n-1}}{2(2\pi i)^{n-1}}\int_{-1}^{1}[\log^{(n-1)}(t+rp) - \log^{(n-1)}(-t+rp)]\,(1-p^2)^{\frac{n-3}{2}}\,dp\ ,$$

if $n \geq 2$, where

$$\Omega_{n-1} = \frac{2(\pi)^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})}\ ,$$

and $r = |x|$. Thus $R^{(n)}$ is a function of t and r alone. Differentiating our formula, we obtain

$$\frac{\partial R^{(n)}}{\partial r} = -\frac{\Omega_{n-1}}{2(2\pi i)^n}\int_{-1}^{1}[\log^{(n)}(t+rp)-\log^{(n)}(-t+rp)]p(1-p^2)^{\frac{n-3}{2}}\,dp.$$

Integrating by parts, we have

$$\frac{\partial R^{(n)}}{\partial r} = -\frac{r\,\Omega_{n-1}}{(n-1)2(2\pi i)^n}\int_{-1}^{1}[\log^{(n+1)}(t+rp)-\log^{(n+1)}(-t+rp)]\times$$
$$\times(1-p_2)^{\frac{n-1}{2}}\,dp\ .$$

Thus we obtain the well-known formula (see [2], p. 408;

$$(4.4)\qquad R^{(n+2)} = -\frac{1}{\pi}\frac{\partial R^{(n)}}{\partial (r^2)}\ .$$

Thus $R^{(n)}$ can be obtained for any n if we know $R^{(2)}$ and $R^{(3)}$.

From (4.2),

$$R^{(2)} = -\frac{1}{2(2\pi i)^2}\frac{\partial}{\partial t}\int_{|\omega|=1}[\log(t+x\cdot\omega)+\log(-t+x\cdot\omega)]\,d\omega\ .$$

Introducing θ as an angular coordinate on the unit circle,

$$R^{(2)} = \frac{\partial}{\partial t}\frac{1}{(2\pi)^2}\int_0^{2\pi}[\log|r\cos\theta+t|+\log|r\cos\theta-t|]\,d\theta\ .$$

This integral can be found in tables of definite integrals. Finally,

$$R^{(2)} = \begin{cases}\dfrac{1}{2\pi}\dfrac{1}{\sqrt{t^2-r^2}} & t^2 > r^2\\[2ex] 0 & t^2 < r^2\end{cases}$$

By definition of the fractional integral,

$$R^{(2)} = \frac{1}{2\sqrt{\pi}}\,\delta^{(-\frac{1}{2})}(t^2-r^2)\ . \tag{4.5}$$

$R^{(3)}$ is given immediately by equation (4.3). We have

$$R^{(3)} = -\frac{2\pi}{2(2\pi i)^3}\int_{-1}^{1}[\log^{(2)}(t+rp)-\log^{(2)}(-t+rp)]\,dp\ .$$

On the real axis,

$$\operatorname{Im}\log Z = \pi\,(1 - H(Z))\ .$$

Hence

$$\operatorname{Im}\log^{(2)} Z = -\pi\,\delta'(Z)\ ,$$

and

$$R^{(3)} = \frac{1}{2(2\pi)^2} \int_{-1}^{1} -\pi[\delta'(t+rp)-\delta'(-t+rp)]\, dp \ .$$

Thus

$$R^{(3)} = \frac{1}{4\pi r} [\delta(t-r)-\delta(t+r)] \ .$$

It follows that

$$(4.6) \qquad R^{(3)} = \frac{1}{2\pi} \delta(t^2-r^2) \ .$$

Using (4.4), (4.5) and (4.6), we have

$$(4.7) \qquad R^{(n)} = \frac{1}{2\pi^{\frac{n-1}{2}}} \delta^{(\frac{n-3}{2})}(t^2-r^2) \ .$$

5. The Method of Stationary Phase

In this section we shall use the method of stationary phase in order to find the singularities of integrals of distributions. Our results are formulated in lemmas 5.1 and 5.2. This method will be given a geometrical interpretation in the following section.

A. Fractional integration and differentiation

First we define fractional integration and differentiation of a distribution which is singular at the origin. We define

$$(5.1) \qquad I_+^{\alpha} f(s) = \frac{1}{\Gamma(\alpha)} \int_{-A}^{s} f(t)\,(s-t)^{\alpha-1}\, dt \ ,$$

$$(5.2) \qquad I_-^{\alpha} f(s) = \frac{1}{\Gamma(\alpha)} \int_{s}^{A} f(t)\,(t-s)^{\alpha-1}\, dt \ .$$

We are only interested in the singular part of $I_{\pm}^{\alpha} f$, and this singular part is independent of A if $|s| < B < A$. Since s will always be restricted to a finite interval, we may formally write $A = \infty$, even though such integrals may not converge. Thus, strictly speaking, we define the singular part of fractional integrals of f by formulas (5.1) and (5.2). In the sequel we shall deal only with singular parts of integrals.

It can be verified in the usual way that

$$I_{+}^{\alpha} I_{+}^{\beta} f = I_{+}^{\alpha+\beta} f ,$$

$$I_{-}^{\alpha} I_{-}^{\beta} f = I_{-}^{\alpha+\beta} f .$$

If we define the transformation T by

$$T f(s) = f(-s) ,$$

then

$$I_{-}^{\alpha} f = T I_{+}^{\alpha} T f .$$

$I_{\pm}^{\alpha}$ may also be defined for negative α by means of differentiation. Note that

$$\frac{d}{ds} I_{-}^{1} f(s) = - f(s) .$$

We shall be especially concerned with the relationship between $I_{+}^{1/2}$ and $I_{-}^{1/2}$. Strictly speaking, these operators do not commute, but their singular parts do commute, i.e.

$$(I_{+}^{1/2} I_{-}^{1/2} - I_{-}^{1/2} I_{+}^{1/2}) f(s)$$

is in C^∞ if $f(s)$ is in C^∞ outside a closed interval. Also, taking singular parts only,

$$I_+^{1/2} I_-^{-1/2} = I_-^{1/2} I_+^{-1/2} = \mathcal{H},$$

where $\mathcal{H}$ is the Hilbert transform:

$$\mathcal{H} f(s) = \int_{-\infty}^{\infty} \frac{f(t)-f(s)}{t-s} dt . \tag{5.3}$$

For future reference, we give some fractional integrals:

$$I_+^\alpha \delta(s) = \begin{cases} \dfrac{s^{\alpha-1}}{\Gamma(\alpha)} & s > 0 \\ 0 & s < 0 , \end{cases} \tag{5.4}$$

$$I_-^\alpha \delta(s) = \begin{cases} 0 & s > 0 \\ \dfrac{(-s)^{\alpha-1}}{\Gamma(\alpha)} & s < 0 . \end{cases} \tag{5.5}$$

If $0 < \alpha < 1$,

$$I_+^\alpha \log^{(1)}(s) = \begin{cases} \Gamma(1-\alpha)\cos \alpha\pi \, s^{\alpha-1} & s > 0 \\ -\Gamma(1-\alpha) \, (-s)^{\alpha-1} & s < 0 , \end{cases} \tag{5.6}$$

$$I_-^\alpha \log^{(1)}(s) = \begin{cases} \Gamma(1-\alpha) \, s^{\alpha-1} & s > 0 \\ -\Gamma(1-\alpha)\cos \alpha\pi (-s)^{\alpha-1} & s < 0 , \end{cases} \tag{5.7}$$

$$(I_-^\alpha + I_+^\alpha) \log^{(1)}(s) = \Gamma(1-\alpha)(1+\cos \alpha\pi) \, (\operatorname{sgn} s) \, |s|^{\alpha-1}, \tag{5.8}$$

$$I_-^{1/2} \log^{(1)}(s) = \pi \, I_+^{1/2} \delta(s) , \tag{5.9}$$

$$I_+^{1/2} \log^{(1)}(s) = -\pi \, I_-^{1/2} \delta(s) . \tag{5.10}$$

B. Two lemmas on integrals of distributions

Now we are in a position to formulate two lemmas regarding integrals of a distribution with respect to a parameter. Let $f(s)$ be a distribution which is smooth except at the origin, and let $\phi(x,\xi)$ be a C^∞ function of x and the parameter ξ. We assume that $(\phi_{x^0},\phi_{x^1},\ldots,\phi_{x^n}) \neq 0$, and that locally ξ is determined as a single-valued function of x by the condition of stationary phase:

$$\phi_\xi(x,\xi) = 0 . \tag{5.11}$$

We write

$$\xi = \zeta(x) ,$$

and

$$\phi(x,\xi) = \phi(x,\zeta(x)) + (\xi-\zeta(x))^\beta \psi(x,\xi) , \tag{5.12}$$

where $\psi(x,\zeta(x)) \neq 0$. Let $\pm$ denote the sign of $\psi(x,\zeta(x))$, and $\mp$ denote its negative. The integer β may depend upon x. We define $\tilde{\phi}(x)$ by

$$\tilde{\phi}(x) \equiv \phi(x,\zeta(x)) . \tag{5.13}$$

Lemma 5.1: If $a(x,\xi)$ is in C^∞, with compact support in ξ, then the integral J, given by

$$J(x) \equiv \int_{-\infty}^{\infty} f(\phi(x,\xi))\, a(x,\xi)\, d\xi ,$$

has the following representation:

(a) If β is even, then

$$(5.14)\quad J = \sum_{j=0}^{N} I_{\mp}^{(j+1)/\beta} f(\tilde{\phi}(x))\, \tilde{a}^{j}(x) + h^{N}(x) ,$$

where

$$\tilde{a}^{0}(x) = \frac{2\Gamma(1/\beta)}{\beta} |\psi(x,\zeta)|^{-1/\beta} a(x,\zeta) .$$

(b) If β is odd, then

$$(5.15)\quad J = \sum_{j=0}^{N} [I_{+}^{(j+1)/\beta} f(\tilde{\phi}(x))\tilde{a}^{+,j}(x) + I_{-}^{(j+1)/\beta} f(\tilde{\phi}(x))\tilde{a}^{-,j}(x)] + h^{N}(x) ,$$

where

$$\tilde{a}^{+,0}(x) = \tilde{a}^{-,0}(x) = \frac{\Gamma(1/\beta)}{\beta} |\psi(x,\zeta)|^{-1/\beta} a(x,\zeta) .$$

(c) If $\phi(x,\xi) \equiv \tilde{\phi}(x)$, i.e. $\beta = \infty$, then it follows at once that

$$J = f(\tilde{\phi}(x))\, \tilde{a}(x) ,$$

where

$$\tilde{a}(x) = \int a(x,\xi)\, d\xi .$$

In any case, $\tilde{a}^{j}(x)$ are in C^{∞}, and the remainder $h^{N}(x)$ can be made as smooth as we like if N is chosen large enough.

Proof: It follows from (5.12) that

$$\phi_{\xi}(x,\xi) = \beta(\xi-\zeta)^{\beta-1}[\psi(x,\xi)+(\xi-\zeta) \text{ reg. funct.}] ,$$

and

$$\frac{1}{\phi_{\xi}(x,\xi)} = \frac{(\xi-\zeta)^{1-\beta}}{\beta\, \psi(x,\zeta)} [1+(\xi-\zeta) \text{ reg.funct.}]$$

Equation (5.12) also implies that

$$(\xi-\zeta) = \left|\frac{\phi-\tilde{\phi}}{\psi(x,\xi)}\right|^{1/\beta} \operatorname{sgn}(\xi-\zeta) ,$$

and hence

$$\frac{1}{\phi_\xi(x,\xi)} = \frac{(\operatorname{sgn}(\xi-\zeta))^{1-\beta}}{\beta\ \psi(x,\zeta)} \left|\frac{\phi-\tilde{\phi}}{\psi(x,\zeta)}\right|^{\frac{1-\beta}{\beta}} [1+(\xi-\zeta) \text{ reg. funct.}]$$

Now, introducing ϕ as the variable of integration, we have

$$J = J_1 + J_2 ,$$

where

$$J_1 = \int_{\phi(x,-\infty)}^{\tilde{\phi}} (\operatorname{sgn}(\xi-\zeta))^{1-\beta} f(\phi) \frac{a_1(x,\xi)}{\beta\ \psi} \left|\frac{\phi-\tilde{\phi}}{\psi}\right|^{\frac{1-\beta}{\beta}} d\phi ,$$

$$J_2 = \int_{\tilde{\phi}}^{\phi(x,\infty)} (\operatorname{sgn}(\xi-\zeta))^{1-\beta} f(\phi) \frac{a_2(x,\xi)}{\beta\ \psi} \left|\frac{\phi-\tilde{\phi}}{\psi}\right|^{\frac{1-\beta}{\beta}} d\phi .$$

Here $a_1(x,\xi)$ and $a_2(x,\xi)$ are regular functions. We note that

$$a_1(x,\zeta) = a_2(x,\zeta) = a(x,\zeta) .$$

Now we assume that β is even and $\psi(x,\zeta) > 0$; the proof for the other cases is obtained by obvious modification. Since $\psi(x,\zeta) > 0$ and β is even, $\phi(x,\infty)$ and $\phi(x,-\infty)$ must both be greater than $\tilde{\phi}$. Nothing is changed if we set $a_1(x,\xi) \equiv 0$ for $\phi \geq \phi(x,-\infty)$, and we set $a_2(x,\xi) \equiv 0$ for $\phi \geq \phi(x,\infty)$. Thus we can extend the limits of integration to the interval $\tilde{\phi} \leq \phi < \infty$. Combining J_1 and J_2,

$$(5.16)\quad J = \int_{\tilde{\phi}}^{\infty} \frac{f(\phi)(\phi-\tilde{\phi})^{(1/\beta-1)}}{\beta(\psi(x,\zeta)^{1/\beta}} (a_1(x,\xi) + a_2(x,\xi))\, d\phi \ .$$

Now we expand the integrand in powers of ϕ :

$$(5.17)\quad \frac{1}{\beta(\psi(x,\zeta))^{1/\beta}} (a_1(x,\xi) + a_2(x,\xi)) =$$

$$= \sum_{j=0}^{N} \frac{\tilde{a}^j(x)(\phi-\tilde{\phi})^{j/\beta}}{\Gamma(\frac{j+1}{\beta})} + H^N(x,\phi)\ (\phi-\tilde{\phi})^{(N+1)/\beta} \ .$$

Substituting (5.17) into (5.16), we obtain

$$J(x) = \sum_{j=0}^{N} I_{-}^{(j+1)/\beta} f(\tilde{\phi})\ \tilde{a}^j(x) + h^N(x) \ ,$$

which was to be proved.

Lemma 5.1 can be used to prove the corresponding result when ξ is a vector: $\xi = (\xi^1,\ldots,\xi^k)$. The most important case is when $\tilde{\phi}(x)$ is a regular envelope of $\phi(x,\xi)$, i.e. when $\beta = 2$. For this case, the more general result can be stated as follows:

Lemma 5.2: Let $\xi = (\xi^1,\ldots,\xi^k)$, and let $f(s)$, $\phi(x,\xi)$, $\zeta(x)$ and $a(x,\xi)$ satisfy the hypotheses of Lemma 5.1. If, in addition

$$\phi_{\xi^i\xi^i}(x,\zeta(x)) < 0 \qquad i = 1,\ldots,\ell,$$

$$\phi_{\xi^i\xi^i}(x,\zeta(x)) > 0 \qquad i = \ell+1,\ldots,k \ ,$$

$$\phi_{\xi^i\xi^j}(x,\zeta(x)) = 0 \qquad i \neq j \ ,$$

then $J(x)$, given by the integral

$$J(x) \equiv \int f(\phi(x,\xi))\, a(x,\xi)\, d\xi ,$$

has the leading term

$$I_-^{(k-\ell)/2}\, I_+^{\ell/2}\, f(\phi(x,\zeta(x)))\, \tilde{a}(x) ,$$

where

$$\tilde{a}(x) = \frac{(2\pi)^{k/2}\, a(x,\zeta(x))}{\left|\prod_{i=1}^{k} \phi_{\xi^i \xi^i}(x,\zeta(x))\right|^{1/2}} .$$

This lemma follows from k applications of Lemma 5.1.

The more general case, where certain of the $\phi_{\xi^i \xi^i}$ vanish, is more delicate. We shall deal with such cases individually.

6. Geometrical Interpretation of the Method of Stationary Phase

In this section we shall give a geometrical interpretation to the method of stationary phase, as applied to our representation formula for the Riemann function. We shall derive the rules given in the introduction, which give the singularities of the Riemann function in terms of the geometry of the ray conoid. Here we consider only equations whose characteristics have uniform multiplicity. The more general case of intersecting sheets of the normal surface will be treated in Section 8.

A. Discussion of Results

We have shown in Section 3[1] that singularities in solutions of linear hyperbolic equations propagate along characteristics, and that the magnitude of the singularity is proportional to the square root of the local ray density. Since the Riemann function is a solution of the partial differential equation, we should expect that the singularities of the Riemann function obey the same law. We shall verify this fact, using the method of stationary phase.

We shall also see that the sign of the determinant which gives the local ray density determines the character

[1] More details are given in D. Ludwig [14].

of the singularity of the Riemann function, since this sign determines the parity of the integer ℓ which appears in Lemma 5.2.

In the case of equations with constant coefficients, the local ray density may be identified with the Gauss curvature of the intersection of the sheet of the ray cone in question and the plane $t = \text{const}$. Thus we are led to the rules given in the introduction, which relate the number of outward pointing principal curvature vectors and the character of the singularity of the Riemann function.

Our results also apply to equations with variable coefficients. We consider "planelike" characteristic surfaces which envelope the ray conoid. These are characteristic surfaces which intersect the initial hyperplane $t = 0$ in an $(n-1)$-dimensional plane. Then instead of the Gauss curvature of the ray conoid, we consider the curvature of a sheet of the ray conoid, relative to the curvature of the planelike characteristic surface which is tangent to the ray conoid at the point in question. This relative curvature is proportional to the relative ray densities within the sheet of the ray conoid and the planelike characteristic. With this modification, our results are valid for equations with variable coefficients: the character of the singularity of the Riemann function is determined by the number of inward pointing directions of relative curvature along certain curves. The magnitude

of the singularity is given by the product of the relative curvatures and the coefficient of the singularity along the planelike characteristic surface. Thus the magnitude of the singularity is proportional to the square root of the local ray density on the ray conoid.

The limiting forms of these expressions near cusps of the ray conoid suggest the rules given in the introduction for determining the singularity of the Riemann function at a cusp. In its present form, the method of stationary phase fails to predict the behavior of the Riemann function at all possible singularities of the ray conoid, but those examples which are amenable to treatment by this method support our conjecture.

It should be noted that our formulas give the contribution to the singularity of the Riemann function from a single point of stationary phase on the unit sphere. The singularity at a particular point on the ray conoid is the sum of the singularities arising from the corresponding points of stationary phase on the unit sphere. Since the point opposite to a stationary point is also a stationary point, all of these formulas should be multiplied by a factor of at least two. In addition, a single point on the ray conoid may correspond to points on different sheets of the normal surface, and these contributions must also be added.

B. <u>Equations with constant coefficients in three independent variables.</u>

First we shall consider equations with constant coefficients, setting $n = 2$. We shall discuss only the leading term in the expansion of the Riemann function, although the other terms can be obtained in a similar fashion. We have an integral of the form

$$(6.1) \qquad J = \int_{|\alpha|=1} f(\phi(t,x,\alpha))\, a(t,x,\alpha)\, d\alpha ,$$

where ϕ is a solution of the characteristic equation:

$$\phi = \lambda(\alpha)t + \alpha \cdot x .$$

Our representation formula (4.1) for the Riemann function shows that $f(\phi) = \log^{(2)} |\phi|$ if the operator is strongly hyperbolic, and $f(\phi) = \log^{(3)}|\phi|$ if $\phi(t,x,\theta)$ is a double root of the characteristic equation, with only one linearly independent right nullvector.

According to the method of stationary phase, J is singular only on the envelope of the surfaces $\phi = 0$, where α runs over the unit circle. Let θ be the angular coordinate on the unit circle. We have a singularity at those points (t,x) where

$$\phi(t,x,\theta_0) = 0 ,$$

$$\phi_\theta(t,x,\theta_0) = 0 ,$$

for some value of θ_0. We assume first that $\phi_{\theta\theta}(t,x,\theta_0) \neq 0$. Let $\pm = \operatorname{sgn} \phi_{\theta\theta}$. Since the singularity of J depends on the integrand only in the neighborhood of (t,x,θ_0), we may use a partition of unity in (6.1) and apply Lemma 5.1, by formally allowing the interval of integration to be $(-\infty,\infty)$. Lemma 5.1 implies that

$$(6.2) \qquad J \sim \frac{\sqrt{2\pi}a(t,x,\theta_0)}{\sqrt{|\phi_{\theta\theta}(t,x,\theta_0)|}} I_{\mp}^{1/2} f(\phi(t,x,\theta_0)) \, .$$

The function $\tilde{\phi}(t,x) = \phi(t,x,\theta_0(t,x))$ is a solution of the characteristic equation, and the vector $a(t,x,\theta_0(t,x))$ is a right nullvector of the corresponding characteristic matrix. The vector $a(t,x,\theta_0(t,x))$ is given by the plane wave solution described in Section 4, with fixed θ. A short calculation shows that $|\phi_{\theta\theta}(t,x,\theta_0(t,x))|^{-1}$ is the curvature of the curve $\tilde{\phi}(t,x) = \text{const.}$ in the plane $t = \text{const.}$, at the point (t,x).

In order to determine the sign of $\phi_{\theta\theta}(t,x,\theta_0(t,x))$, we must examine the curvature vector of the intersection of the ray surface with the plane $t = \text{const.}$ Let us first assume that $\alpha(\theta_0) \cdot x_0 > 0$, i.e. $\phi_t = \lambda(\theta_0) < 0$. Then if the curvature vector points toward the origin at x_0 (Fig. 8), we have

$$\phi(t,x,\theta_0(x)) \geq \phi(t,x,\theta_0(x_0)) \, , \quad \text{near } x_0 \, .$$

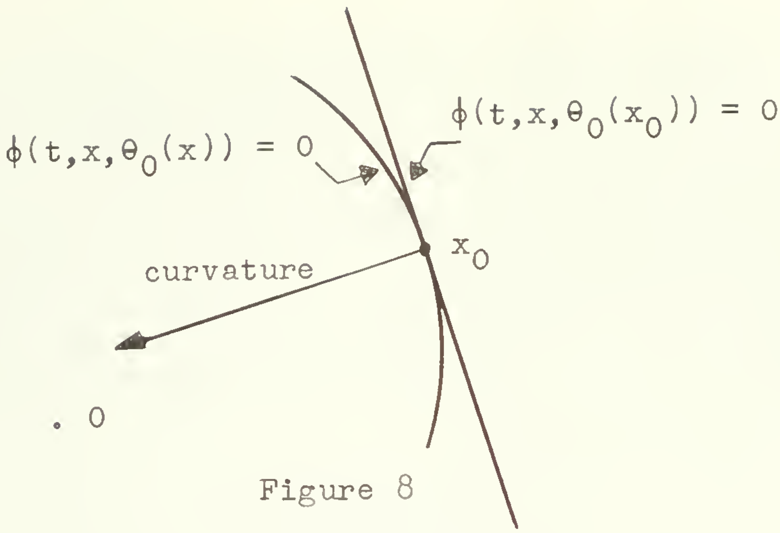

Figure 8

Hence we must have $\phi_{\theta\theta}(t,x,\theta_0(x)) < 0$, and

$$\operatorname{sgn} \phi_{\theta\theta}(t,x,\theta_0(x)) = \operatorname{sgn} \lambda(\theta_0(x)) .$$

By similar reasoning for the remaining cases we find:

$$(6.3) \qquad \operatorname{sgn} \phi_{\theta\theta}(t,x,\theta_0(x)) = \operatorname{sgn} \lambda(\theta_0(x)) ,$$

if the curvature vector points inwards,

$$(6.4) \qquad \operatorname{sgn} \phi_{\theta\theta}(t,x,\theta_0(x)) = - \operatorname{sgn} \lambda(\theta_0(x)) ,$$

if the curvature vector points outwards.

Now we can combine our results and use Lemma 5.1.

Let $\pm = \operatorname{sgn} \lambda(\theta_0)$. If the curvature vector points inwards,

$$R \sim \frac{\sqrt{2\pi}}{\sqrt{|\phi_{\theta\theta}|}} a(t,x,\theta_0)\, I_{\mp}^{1/2} \log^{(2)}(\tilde{\phi}) .$$

Using (5.9) and (5.10),

$$(6.5) \qquad R \sim \frac{\sqrt{2\pi}\,\pi}{\sqrt{|\phi_{\theta\theta}|}} a(t,x,\theta_0)\, I_{+}^{-1/2} \delta(\pm\tilde{\phi}) ,$$

or

$$(6.6) \quad R \sim \frac{\pi}{\sqrt{2}}\, a(t,x,\theta_0)\, |K(t,x)|^{-1/2}\, (\pm\tilde{\phi})^{-3/2}$$

Here $K(t,x)$ is the curvature of the intersection of the ray cone and the plane $t = \text{const.}$, and the distribution $(\pm\tilde{\phi})^{-3/2}$ vanishes if $\pm\tilde{\phi} < 0$.

In a similar fashion we find that if the curvature vector points outwards, then

$$(6.7) \quad R \sim -\frac{\pi}{\sqrt{2}}\, a(t,x,\theta_0)\, |K(t,x)|^{-1/2}\, (\mp\tilde{\phi})^{-3/2} .$$

If $\phi_{\theta\theta}(t,x,\theta_0) = 0$, but $\phi_{\theta\theta\theta}(t,x,\theta_0) \neq 0$, then we obtain from (5.15) and (5.8) that

$$(6.8) \quad R \sim \frac{6^{1/3}}{3}\, \frac{\Gamma(1/3)\Gamma(2/3)}{|\phi_{\theta\theta\theta}(t,x,\theta_0)|^{1/3}}\, a(t,x,\theta_0)\, |\tilde{\phi}|^{-5/3} .$$

These formulas agree with our results for the wave equation, and they imply our results for two-dimensional propagation of hydromagnetic disturbances given in Section 2.

C. Equations with constant coefficients in more than three independent variables

We can extend our results to apply to any number of independent variables, using the results for three independent variables. Our aim is to apply the method of stationary phase to an integral of the form

$$J = \int_{|\alpha|=1} f(\phi(t,x,\alpha))\, a(t,x,\alpha)\, d\alpha ,$$

where $\phi(t,x,\alpha) = \lambda(\alpha)t + \alpha \cdot x$.

As before, the singularity of J is determined by the points of stationary phase:

$$\phi_\alpha(t,x,\alpha_0) = 0 .$$

Here ϕ_α denotes the gradient of ϕ with respect to α, and α is restricted by the condition that $|\alpha| = 1$. We can introduce orthogonal coordinates $\xi^1,\dots,\xi^{n-1}$ on the unit sphere such that the matrix

$$(6.9) \qquad H = (\phi_{\xi^i\xi^j}(t,x,\alpha_0)) \qquad (i,j=1,\dots,n-1),$$

is diagonal. Here we consider $\phi(t,x,\alpha(\xi))$ as a function of ξ.

First we assume that H is non-singular. Then $\xi^1,\dots,\xi^{n-1}$ and

$$(6.10) \qquad \tilde{\phi}(t,x) = \phi(t,x;\xi(t,x))$$

are determined as functions of $x^1,\dots,x^n$ for fixed t, and conversely $x^1,\dots,x^n$ are determined as functions of $\xi^1,\dots,\xi^{n-1}$ and $\tilde{\phi}$. We write $\xi^i = \zeta^i(t,x)$ $(i = 1,\dots,n-1)$. From the condition of stationary phase:

$$(6.11) \qquad \frac{\partial\phi(t,x;\zeta(t,x))}{\partial\xi^i} = 0 \qquad i = 1,\dots,n-1,$$

we have

$$(6.12) \qquad \phi_{\xi^i x^j} + \sum_{k=1}^{n-1} \phi_{\xi^i\xi^k}\frac{\partial\zeta^k}{\partial x^j} = 0 \quad (i=1,\dots n-1; j=1,\dots,n).$$

Hence, since H is diagonal,

(6.13) $$\frac{\partial(\tilde{\phi},\zeta)}{\partial(x)} = \frac{(-1)^{n-1}\det(U)}{\det(H)},$$

where

(6.14) $$U = \begin{pmatrix} \tilde{\phi}_{x^1} & \cdots & \tilde{\phi}_{x^n} \\ \phi_{\xi^1 x^1} & \cdots & \phi_{\xi^1 x^n} \\ \vdots & & \\ \phi_{\xi^{n-1} x^1} & \cdots & \phi_{\xi^{n-1} x^n} \end{pmatrix}.$$

Since $\phi(t,x;\xi) = \lambda(\xi)t + x\cdot\alpha(\xi)$, it is easy to see that $|\det(U)| = 1$. Therefore

(6.15) $$\left|\frac{\partial(\tilde{\phi},\zeta)}{\partial(x)}\right| = \left|\frac{1}{\det(H)}\right| .$$

Since $\alpha(\zeta)$ is the normal to $\phi(t,x,\zeta) = \text{const.}$, it follows that $\partial(\tilde{\phi},\zeta)/\partial(x)$ is just the ratio of the area of the spherical image of a sheet of the ray cone to the corresponding area on the ray cone. Hence,

(6.16) $$\left|\frac{1}{\det(H)}\right| = |K(t,x)|\,|\phi_x| = |K(t,x)| ,$$

where $K(t,x)$ is the Gauss curvature of the intersection of a sheet of the ray cone and the plane $t = \text{const.}$

In order to determine the signs of $\phi_{\xi^i\xi^i}(t,x;\zeta)$ $(i = 1,\ldots,n-1)$, we may use the results for $n = 2$, since the coordinate curves $\zeta^i = \text{const.}$ $(i \neq j)$ are mapped onto the lines of curvature of the ray cone. We shall demonstrate this fact in our treatment of equations with variable coefficients. Thus we find that

$$\operatorname{sgn} \phi_{\xi^i\xi^i} = \operatorname{sgn} \lambda(\alpha_0) \text{ if the corresponding curvature vector points inwards,}$$

$$\operatorname{sgn} \phi_{\xi^i\xi^i} = -\operatorname{sgn} \lambda(\alpha_0) \text{ if the corresponding curvature vector points outwards.}$$

Again, we let $\pm = \operatorname{sgn} \lambda(\alpha_0)$. Let m be the number of outward pointing curvature vectors. For strongly hyperbolic equations, (4.1) states that

$$(4.1) \quad R \sim \int_{|\alpha|=1} \log^{(n)}(\phi(t,x,\alpha))\, a(t,x,\alpha)\, d\alpha \, .$$

The imaginary part of this expression vanishes. Applying Lemma 5.2 to this expression, we find that if n is even and m is even,

$$R \sim I_{\mp}^{1/2}\, I_{\mp}^{\frac{n-m-2}{2}}\, I_{\pm}^{\frac{m}{2}} \log^{(n)}(\tilde{\phi})\, [(2\pi)^{\frac{n-1}{2}} |K(t,x)|\, a(t,x,\alpha_0)].$$

Since $(n-m-2)/2$ is an integer, we have

$$R \sim i^{n-m-2}\, I_{\pm}^{\frac{n-2}{2}}\, I_{\mp}^{1/2} \log^{(n)}(\tilde{\phi})[(2\pi)^{\frac{n-1}{2}} |K(t,x)|\, a(t,x,\alpha_0)].$$

Using (5.9) and (5.10), we have

$$R \sim \pm\, \pi\, i^{n-m-2}\, I_{\pm}^{\frac{n-1}{2}}\, \delta^{(n-1)}(\tilde{\phi})\, [(2\pi)^{\frac{n-1}{2}} |K(t,x)|\, a(t,x,\alpha_0)],$$

or

$$(6.17) \quad R \sim I_{+}^{\frac{1-n}{2}}\, \delta(\pm\tilde{\phi})\, [i^{n-m-2}\, \pi(2\pi)^{\frac{n-1}{2}} |K(t,x)|\, a(t,x,\alpha_0)].$$

In a similar fashion, we find that if n is even and m is odd,

$$(6.18) \quad R \sim I_+^{\frac{1-n}{2}} \, \delta(\mp\tilde{\phi}) \, [i^{m-1} \, \pi(2\pi)^{\frac{n-1}{2}} \, |K(t,x)| \, a(t,x,\alpha_0)].$$

If n is odd, the real part of R is obtained by taking the imaginary part of $\log^{(n)}$, since $a(t,x,\alpha)$ is pure imaginary. Since

$$\log s = \log |s| + i \, \pi(1-H(s)) \, ,$$

we have

$$R \sim \frac{\pi}{i} \, I_{\mp}^{\frac{n-1-m}{2}} \, I_{\pm}^{m/2} \, \delta^{(n-1)}(\tilde{\phi}) \, [(2\pi)^{\frac{n-1}{2}} \, |K(t,x)| \, a(t,x,\alpha_0)].$$

If m is even, this expression simplifies to

$$(6.19) \quad R \sim I_+^{\frac{1-n}{2}} \, \delta(\pm\tilde{\phi}) \, [i^{n-m-2}\pi(2\pi)^{\frac{n-1}{2}} \, |K(t,x)| \, a(t,x,\alpha_0)],$$

which is identical to (6.17). Thus if m is even, then the singularity of R is given by the same formal expression for even and odd n.

If m is odd, then

$$R \sim \frac{\pi}{i} \, I_{\mp}^{\frac{n-1-m}{2}} \, I_{\pm}^{m/2} \, \delta^{(n-1)}(\tilde{\phi}) \, [(2\pi)^{\frac{n-1}{2}} \, |K(t,x)| \, a(t,x,\alpha_0)].$$

Using (5.9) and (5.10), this expression simplifies to

$$(6.20) \quad R \sim \pm \, I_+^{\frac{-1-n}{2}} \, \log |\tilde{\phi}| \, [i^{n-m-1}(2\pi)^{\frac{n-1}{2}} \, |K(t,x)| \, a(t,x,\alpha_0)].$$

Notice that R has a logarithmic singularity only if n and m are odd. Formulas (6.17-6.20) justify the rules given in the introduction, for the case where the ray cone is regular.

In order to illustrate the procedure where the ray cone

is singular, i.e. where the normal surface has a point of inflection or a point at infinity, we consider the equations of hydromagnetics. We have already discussed two-dimensional propagation. For three-dimensional propagation, the analogous procedure may be used. Referring to Figures 2 and 3 of Section 2, we see that on the circular cusps of the ray surface, one of the principal curvatures is infinite, while the other is finite. Thus we may apply Lemma 5.1 twice; first with $\beta = 2$, and then with $\beta = 3$. We conclude that

$$(6.21)\quad R \sim (I_+^{1/3}+I_-^{1/3})\, I_{\pm}^{1/2}\, \delta^{(2)}(\tilde{\phi}) \times$$

$$\times[\text{const. } |\phi_{\xi^1\xi^1}|^{-1/2}\, |\phi_{\xi^2\xi^2\xi^2}|^{-1/3}\, a(t,x,\alpha_0)].$$

Near the ray cone, R increases like $|\tilde{\phi}|^{-13/6}$.

Strictly speaking, our method does not apply to the singularities on the inner cusps of the ray surface on the axis of symmetry, or at other points on the axis of symmetry. The reason is that these points correspond to isolated multiple points of the normal surface. Near these points of the normal surface, the associated right and left nullvectors of the characteristic matrices form a singular vector field. In particular, these vectors have different limits at the multiple points, depending upon the direction of approach. This is a degenerate kind of conical singularity of the normal surface. Consequently the vector $a(t,x,\alpha_0)$ is singular in the neighborhood of these points. Nevertheless,

the limiting forms of our expressions (6.19) and (6.20) suggest the behavior of R given in Section 2.

D. Equations with variable coefficients

We shall treat equations with variable coefficients by means similar to those used for equations with constant coefficients. The method of stationary phase can be applied as before. The only new difficulty arises in the interpretation of the numbers $\phi_{\xi^i \xi^i}(t,x;z,\alpha_0)$ $(i = 1,\ldots,n-1)$, which appear in Lemma 5.2.

We recall that, for strongly hyperbolic equations, equation (4.1) states that

$$(4.1) \quad R \sim \int_{|\alpha|=1} \log^{(n)}(\phi(t,x;z,\alpha))\, a(t,x;z,\alpha)\, d\alpha \; .$$

Again, the singular part of R comes from the points of stationary phase:

$$(6.22) \qquad \phi_\alpha(t,x;z,\alpha_0) = 0 \quad ,$$

where ϕ_α denotes the gradient of ϕ taken on the unit sphere.

First we shall deal with the "planelike" characteristic surfaces, given by the equation

$$(6.23) \qquad \phi(t,x;z,\alpha) = 0 \quad .$$

Here z and α are regarded as fixed parameters. It will again be convenient to introduce $\xi^1,\ldots,\xi^{n-1}$ as orthogonal

coordinates on the unit sphere in the neighborhood of $\alpha_0(t,x,z)$. The function ϕ is a solution of the characteristic equation

$$\phi_t = \lambda(t,x,\phi_x) \quad , \tag{6.24}$$

with the initial condition

$$\phi(0,x;z,\alpha) = (x-z)\cdot\alpha \quad . \tag{6.25}$$

As before in Section 3, we shall solve (6.24) and (6.25) by means of ordinary differential equations along rays. In the initial plane $t = 0$, we introduce coordinates $y^1,\ldots,y^n$ by means of the relations

$$\begin{cases} y^n = (x-z)\cdot\alpha \\ y^1 = (x-z)\cdot\alpha_{\xi^1} \\ \vdots \\ y^{n-1} = (x-z)\cdot\alpha_{\xi^{n-1}} \end{cases} \quad . \tag{6.26}$$

We introduce the parameter s by means of the equations

$$\begin{cases} \dot{t}(s,y) = 1 & \\ \dot{x}^i(s,y) = -\dfrac{\partial\lambda(t,x,p)}{\partial p_i} & i = 1,\ldots,n \\ \dot{p}_0(s,y) = \dfrac{\partial\lambda}{\partial t}(t,x,p) & \\ \dot{p}_i(s,y) = \dfrac{\partial\lambda}{\partial x^i}(t,x,p) & i = 1,\ldots,n \ , \end{cases} \tag{6.27}$$

with the initial conditions

$$(6.28)\qquad \begin{cases} t(0,y) = 0 \\ x^i(0,y) = X^i(y) & i = 1,\dots,n \\ p(0,y) = \alpha \\ p_0(0,y) = \lambda(0,X(y),\alpha) . \end{cases}$$

Here $x = X(y)$ is determined from (6.26). Finally, we determine y as a function of x and t from the equations

$$(6.29)\qquad \begin{cases} x^i = x^i(s,y) & i = 1,\dots,n , \\ t = s . \end{cases}$$

Then

$$(6.30)\qquad \phi(t,x;z,\alpha) = y^n(t,x) .$$

Now we shall show that

$$(6.31)\qquad \phi_{\xi^i}(t,x;z,\alpha) = y^i(t,x) \qquad i = 1,\dots,n-1 .$$

Equations (6.25) and (6.26) show that (6.31) is satisfied for $t = 0$. It only remains to show that ϕ_{ξ^i} is constant along each ray, since the rays are defined by the equations $y^i = \text{const.}$ Differentiating (6.24) with respect to ξ^i, we obtain

$$\phi_{\xi^i t} = \sum_{j=1}^{n} \lambda p_j \, \phi_{\xi^i x^j} ,$$

or

$$(6.32)\qquad \phi_{\xi^i t}\, \dot{t} + \sum_{j=1}^{n} \phi_{\xi^i x^j}\, \dot{x}^j = 0 .$$

Hence ϕ_{ξ^i} is constant along a ray, and (6.31) is proved.

In Section 3, the local ray density on the surface $\phi = \text{const.}$ was defined by the ratio $\partial(\phi,y)/\partial(x)$. Equation (6.31) implies that

$$\frac{\partial(\phi,y)}{\partial(x)} = \det(U) \;, \tag{6.33}$$

where

$$U = \begin{pmatrix} \phi_{x^1} & \cdots & \phi_{x^n} \\ \phi_{\xi^1 x^1} & \cdots & \phi_{\xi^1 x^n} \\ \vdots & & \\ \phi_{\xi^{n-1} x^1} & \cdots & \phi_{\xi^{n-1} x^n} \end{pmatrix} . \tag{6.34}$$

Now, following our argument for equations with constant coefficients we shall analyze the relationship between the planelike surfaces and the ray conoid. We determine $\xi^1,\ldots,\xi^{n-1}$ as functions of t,x,z from the condition of stationary phase:

$$\phi_{\xi^i}(t,x;z,\zeta(t,x,z)) = 0 \qquad i = 1,\ldots,n-1 \,. \tag{6.35}$$

At regular points of the ray conoid, this is certainly possible: we have

$$\phi_{\xi^i x^j} + \sum_{k=1}^{n-1} \phi_{\xi^i \xi^k} \frac{\partial \zeta^k}{\partial x^j} = 0 \qquad i=1,\ldots n-1; j=1,\ldots n \,. \tag{6.36}$$

By a rotation of the coordinates $\xi^1,\ldots,\xi^{n-1}$, we may insure that the Hessian matrix

$$H = \begin{pmatrix} \phi_{\xi^1 \xi^1} & \cdots & \phi_{\xi^1 \xi^{n-1}} \\ \vdots & & \\ \phi_{\xi^{n-1} \xi^1} & \cdots & \phi_{\xi^{n-1} \xi^{n-1}} \end{pmatrix} \tag{6.37}$$

is diagonal. At regular points of the ray conoid, H is non-singular. Then (6.36) immediately implies that

$$(6.38) \qquad \frac{\partial(\tilde{\phi},\zeta)}{\partial(x)} = (-1)^{n-1} \frac{\det(U)}{\det(H)} ,$$

where, as before

$$(6.39) \qquad \tilde{\phi}(t,x,;z) = \phi(t,x;z,\zeta(t,x,z)) .$$

It follows immediately from (6.36) and (6.32) that $\zeta^i(t,x,z)$ is constant along rays; hence $\partial(\tilde{\phi},\zeta)/\partial(x)$ is proportional to the local ray density on the ray conoid. Then (6.38) and (6.33) imply that $(\det(H))^{-1}$ is equal to the ray density on the ray conoid divided by the ray density on the planelike characteristic which is tangent to the ray conoid. This fact, together with Lemma 5.2, proves our earlier remark that the coefficient of the singularity of R must be proportional to the square root of the local ray density of the ray conoid. The factor of proportionality depends only upon the characteristic strip tangent to the ray conoid.

Now we shall interpret the numbers $\phi_{\xi^i\xi^i}(t,x;z,\zeta(t,x,z))$ in terms of the relative curvatures of the ray conoid and the planelike characteristic surface. Let S be the intersection of a sheet of the ray conoid with a hyperplane $t = \text{const.}$, and let P be the corresponding intersection of the planelike characteristic surface which is tangent to the ray conoid at a point of S. The equation of S is

(6.40) $\quad \tilde{\phi}(t,x;z) = \phi(t,x;z,\zeta(t,x,z)) = 0 \quad .$

If t_0, x_0 is the point of tangency in question, the equation of P is

(6.41) $$\phi(t,x;z,\zeta(t_0,x_0;z)) = 0 \quad .$$

We introduce $\phi(t,x;z,\zeta(t_0,x_0;z))$ and $\zeta^i(t,x;z)$ as new coordinates in x-space. In terms of these new coordinates, the planelike surfaces $\phi(t,x;z,\xi) = \text{const.}$ are given by the equation

(6.42) $$\psi(t,\phi,\zeta;z,\xi) = \phi(t,x(\phi,\zeta);z,\xi) = \text{const.}$$

The surface S is given by the equation

(6.43) $$\tilde{\psi}(t,\phi,\zeta;z) = \psi(t,\phi,\zeta;z,\zeta) = 0 \quad .$$

It follows from (6.35) and (6.42) that

(6.44) $$\psi_{\xi^i}(t,\phi,\zeta;z,\zeta) = 0 \qquad i = 1,\ldots,n-1 \; .$$

Hence, differentiating with respect to ζ^j :

(6.45) $$\psi_{\xi^i\zeta^j}(t,\phi,\zeta;z,\zeta) + \psi_{\xi^i\xi^j}(t,\phi,\zeta;z,\zeta) = 0 \qquad i,j = 1,\ldots,n-1 \; .$$

It follows from (6.43) and (6.44) that

(6.46) $$\tilde{\psi}_{\zeta^i}(t,\phi,\zeta;z) = \psi_{\zeta^i}(t,\phi,\zeta;z,\zeta) \qquad i = 1,\ldots,n-1 \; .$$

Differentiating (6.46), we obtain

$$\tilde{\psi}_{\zeta^i\zeta^j}(t,\phi,\zeta;z) = \psi_{\zeta^i\zeta^j}(t,\phi,\zeta;z,\zeta) + \psi_{\zeta^i\xi^j}(t,\phi,\zeta;z,\zeta)$$

$$i,j = 1,\ldots,n-1$$

or, using (6.45),

$$\tilde{\psi}_{\zeta^i\zeta^j}(t,\phi,\zeta;z) = \psi_{\zeta^i\zeta^j}(t,\phi,\zeta;z,\zeta) - \psi_{\xi^i\xi^j}(t,\phi,\zeta;z,\zeta) \tag{6.47}$$

$$i,j = 1,\ldots,n-1 \ .$$

It follows immediately from (6.42) that $\psi_{\xi^i\xi^j} = \phi_{\xi^i\xi^j}$. Hence the (diagonal) Hessian matrix given in (6.37) represents the relative curvatures of S and P, measured along the (n-1) coordinate lines $\zeta^i = \text{const.}$ $(i \neq j)$. If the differential operator has constant coefficients, then the curvature of P is zero, and $\zeta^1,\ldots,\zeta^{n-1}$ from an orthogonal system. Equation (6.47) states in this case that our coordinate lines are the lines of principal curvature of S.

Now, applying the method of stationary phase as we did for equations with constant coefficients, we obtain (6.17-6.20), where $\pm = \operatorname{sgn} \phi_t(t,x;z,\zeta)$ and $|K(t,x)| = |\det(H)|^{-1}$. Singular points on the ray conoid may be treated in a similar fashion.

7. Uniform Expansions near Intersections of the Normal Surface.

This section will be concerned with preliminary calculations, directed towards a construction of the Riemann function for equations which have self-intersections of the normal surface. When two roots of the characteristic equation approach each other, then in general the expansion of the singular part of the solution changes character, i.e. a "Stokes phenomenon" appears.

We shall attack the simplest problems of this sort by a method which seems to have wide applicability.[1] The method is based upon the belief that the form of the asymptotic expansion of the solution of a differential equation depends only upon the geometry of the relevant characteristics. It follows that once we have an asymptotic expansion of the solution of a (simple) problem, then the form of this expansion will be valid for all problems with a similar geometry of characteristics.

In the present case, consideration of a simple prototype leads to expansions with a typical term of the form

$$\int_0^t f\left(\phi^1 + \frac{\phi^2-\phi^1}{t}\tau\right) g(t,x;\tau)\, d\tau \,.$$

[1] This method is frequently used in dealing with asymptotic expansions. See, for example, J. B. Keller [8] and R. E. Langer [9].

Where $\phi^1 \neq \phi^2$, this expression is analogous to the expansions of Section 3, with wave forms $f_1(\phi^1)$ and $f_1(\phi^2)$. As before, f_1 is the indefinite integral of f. However, where $\phi^1 = \phi^2$ the wave form is given by $f(\phi^1)$.

A. A simple example

We consider the equation

$$(7.1) \qquad (\frac{\partial}{\partial t} - \frac{\partial}{\partial x})(\frac{\partial}{\partial t} - \lambda \frac{\partial}{\partial x}) u = 0 ,$$

or the equivalent first-order system:

$$(7.2) \qquad \begin{cases} (\frac{\partial}{\partial t} - \lambda \frac{\partial}{\partial x}) u = v \\ \\ (\frac{\partial}{\partial t} - \frac{\partial}{\partial x}) v = 0 . \end{cases}$$

Here we assume that λ depends upon certain parameters; in particular $\lambda = 1$ for certain values of these parameters. We prescribe Cauchy data of the form

$$(7.3) \qquad \begin{aligned} u(0,x) &= 0 \\ v(0,x) &= \sum_{j=0}^{\infty} f_j(x)\, b^k(x) , \end{aligned}$$

where, as before, $\frac{d}{dx} f_j(x) = f_{j-1}(x)$ $(j = 1,2,\ldots)$.

From Duhamel's principle we find that

$$v(t,x) = \sum_{j=0}^{\infty} f_j(t+x)\, a^j(t,x) ,$$

and

$$u(t,x) = \int_0^t w(t,x;\tau)\, d\tau ,$$

where

$$\left(\frac{\partial}{\partial t} - \lambda \frac{\partial}{\partial x}\right) w(t,x;\tau) = 0$$

$$w(\tau,x;\tau) = v(\tau,x) .$$

Hence w has the form

$$w(t,x;\tau) = \sum_{j=0}^{\infty} f_k(\lambda(t-\tau) + \tau + x)\, g^j(t,x;\tau) ,$$

and

$$(7.4) \quad u(t,x) = \sum_{j=0}^{\infty} \int_0^t f_j(\lambda t+x+(1-\lambda)\tau)\, g^j(t,x;\tau)\, d\tau .$$

This expansion for u exhibits the behavior described in the introduction to this section. We have only to observe that two solutions of the characteristic equation are

$$\phi^1(t,x) = \lambda t + x ,$$

$$\phi^2(t,x) = t + x ,$$

and

$$1-\lambda = \frac{\phi^2-\phi^1}{t} .$$

B. Strongly hyperbolic systems.

Now we consider a general first-order system with constant coefficients:

$$(7.5) \qquad \mathcal{L}u \equiv u_t + \sum_{i=1}^{n} A^i \frac{\partial u}{\partial x^i} + Bu = 0 .$$

We permit two roots of the characteristic equation to coincide for certain directions of the wave normal, but require that the corresponding characteristic nullvectors vary smoothly and remain linearly independent.

Guided by the solution (7.4), we look for solutions of (7.5) of the form

$$(7.6)\quad u(t,x;\alpha) = \sum_{j=0}^{\infty} f_j(\phi)\, a^j(t,x;\alpha) + \sum_{j=0}^{\infty} f_j(\psi) b^j(t,x;\alpha) + \sum_{j=0}^{\infty} \int_0^t f_j(\phi+\rho\tau)\, g^j(t-\tau,x;\alpha,\tau)\, d\tau + \sum_{j=0}^{\infty} \int_0^t f_j(\psi-\rho s)\, h^j(t-s,x;\alpha,s)\, ds \quad .$$

Here $\phi(t,x;\alpha)$ and $\psi(t,x;\alpha)$ are "plane wave" solutions of the characteristic equation:

$$(7.7)\qquad \phi(t,x;\alpha) = \lambda^1(\alpha)t + \alpha\cdot x \quad ,$$

$$(7.8)\qquad \psi(t,x;\alpha) = \lambda^2(\alpha)t + \alpha\cdot x \quad ,$$

and

$$(7.9)\qquad \rho(\alpha) = \lambda^2(\alpha) - \lambda^1(\alpha) \quad .$$

We make the simplifying assumption that the quantities a^j, b^j, g^j, h^j do not depend upon x or α. The construction of Section 4 shows that the Riemann function can be represented in terms of such solutions of (7.5).

Let

$$(7.10) \qquad A^{\phi} = \lambda^1(\alpha)\, I + \sum_{i=1}^{n} A^i \alpha_i \quad ,$$

$$(7.11) \qquad A^{\psi} = \lambda^2(\alpha)\, I + \sum_{i=1}^{n} A^i \alpha_i \quad ;$$

thus A^{ϕ} and A^{ψ} are the characteristic matrices corresponding to ϕ and ψ, respectively. Our assumptions about $\mathcal{L}$ imply that vectors R^{ϕ}, R^{ψ}, L^{ϕ} and L^{ψ} exist, which depend smoothly on α, such that

$$(7.12) \begin{cases} A^{\phi}R^{\phi} = 0 \,, & L^{\phi}A^{\phi} = 0 \,, \quad L^{\phi}\cdot R^{\phi} = 1 \,, \\ A^{\psi}R^{\psi} = 0 \,, & L^{\psi}A^{\psi} = 0 \,, \quad L^{\psi}\cdot R^{\psi} = 1 \,, \\ L^{\phi}\cdot R^{\psi} = 0 \,, & L^{\psi}\cdot R^{\phi} = 0 \,. \end{cases}$$

Now we apply (7.5) to (7.6), and collect terms:

$$\begin{aligned} (7.13) \quad 0 = \mathcal{L}u = {} & f_{-1}(\phi)[A^{\phi}a^0] + f_{-1}(\psi)\,[A^{\psi}b^0] \\ & + \sum_{j=0}^{\infty} f_j(\phi)\,[A^{\phi}a^{j+1} + \mathcal{L}(a^j) + h^j(0;t)] \\ & + \sum_{j=0}^{\infty} f_j(\psi)\,[A^{\psi}b^{j+1} + \mathcal{L}(b^j) + g^j(0;t)] \\ & + \int_0^t f_{-1}(\phi+\rho\tau)\, A^{\phi}\, g^0(t-\tau;\tau)\, d\tau \\ & + \int_0^t f_{-1}(\psi-\rho s)\, A^{\psi}\, h^0(t-s;s)\, ds \\ & + \sum_{j=0}^{\infty}\int_0^t [A^{\phi}g^{j+1}(t-\tau;\tau) + \mathcal{L}g^j(t-\tau;\tau)]\, f_j(\phi+\rho\tau)\, d\tau \\ & + \sum_{j=0}^{\infty}\int_0^t [A^{\psi}h^{j+1}(t-s;s) + \mathcal{L}h^j(t-s;s)]\, f_j(\psi-\rho s)\, ds \,. \end{aligned}$$

Thus, in analogy to the procedure of Section 3, we are led to the conditions:

$$(7.14)\qquad A^{\phi}\, a^{0}(t) = 0 ,$$

$$(7.15)\qquad A^{\psi}\, b^{0}(t) = 0 ,$$

$$(7.16)\qquad A^{\phi} a^{j+1}(t) + \mathcal{L} a^{j}(t) + h^{j}(0;t) = 0 \quad (j = 0,1,\ldots),$$

$$(7.17)\qquad A^{\psi} b^{j+1}(t) + \mathcal{L} b^{j}(t) + g^{j}(0;t) = 0 \quad (j = 0,1,\ldots),$$

$$(7.18)\qquad A^{\phi}\, g^{0}(t-\tau;\tau) = 0 ,$$

$$(7.19)\qquad A^{\psi}\, h^{0}(t-\tau;\tau) = 0 ,$$

$$(7.20)\qquad \int_0^t f_j(\phi+\rho\tau)\, [A^{\phi} g^{j+1}(t-\tau;\tau) + \mathcal{L} g^{j}(t-\tau;\tau)]\, d\tau$$

$$+ \int_0^t f_j(\psi-\rho s)[A^{\psi} h^{j+1}(t-s;s) + \mathcal{L} h^{j}(t-s;s)]\, ds = 0 ,$$

$$(j = 0,1,\ldots).$$

It should be noted that the essential difficulty (which we shall presently overcome) is that the inverses of A^{ϕ} and A^{ψ} do not depend continuously on α ; when $\rho(\alpha)$ is small it is not sufficient to require that $L^{\phi} z = 0$ or $L^{\psi} z = 0$ in order to have a bounded solution to

$$A^{\phi}\, y = z ,$$

or

$$A^{\psi}\, y = z ,$$

respectively. However, if we require that $L^{\phi} z = L^{\psi} z = 0$, then both equations have bounded solutions.

Our equations (7.16) and (7.17) thus can be solved if we require that

$$(7.21)\quad \begin{cases} L^{\phi}[\mathcal{L}a^j + h^j(0;t)] = 0 , \\ \\ L^{\psi}[\mathcal{L}a^j + h^j(0;t)] = 0 , \end{cases}$$

$$(7.22)\quad \begin{cases} L^{\phi}[\mathcal{L}b^j + g^j(0;t)] = 0 , \\ \\ L^{\psi}[\mathcal{L}b^j + g^j(0;t)] = 0 . \end{cases}$$

The integrals in (7.20) can be treated in a similar fashion if we set $s + \tau = t$; then we obtain

$$(7.23)\quad \int_0^t f_j(\phi+\rho\tau)\,[A^{\phi}g^{j+1}(t-\tau;\tau) + \mathcal{L}g^j(t-\tau;\tau)$$
$$+ A^{\psi}h^{j+1}(\tau;t-\tau) + \mathcal{L}h^j(\tau;t-\tau)]\,d\tau = 0 .$$

It should be noted that the operator $\partial/\partial t$ acts on the first argument of $h^j(\tau;t-\tau)$. Thus we are led to the condition

$$(7.24)\quad A^{\phi}g^{j+1}(t-\tau;\tau) + \mathcal{L}g^j(t-\tau;\tau) + A^{\psi}h^{j+1}(\tau;t-\tau)$$
$$+ \mathcal{L}h^j(\tau;t-\tau) = 0 .$$

There is no loss of generality if we set

$$(7.25)\qquad A^{\psi}h^{j+1}(\tau;t-\tau) = 0 .$$

In order to solve (7.24) for g^{j+1}, we impose the conditions

$$L^{\phi}\,[\mathcal{L} g^{j}(t-\tau;\tau) + \mathcal{L} h^{j}(\tau;t-\tau)] = 0 \ ,$$

(7.26)

$$L^{\psi}\,[\mathcal{L} g^{j}(t-\tau;\tau) + \mathcal{L} h^{j}(\tau;t-\tau)] = 0 \ .$$

Now we may solve (7.14-7.19), (7.21), (7.22) (7.24-7.26) in a recursive fashion. We shall show how the first terms in (7.6) are obtained; the others may be obtained in a similar manner. From (7.14) and (7.15), we have

$$a^{0} = \sigma_1 R^{\phi} \ , \tag{7.27}$$

$$b^{0} = \sigma_2 R^{\psi} \ ; \tag{7.28}$$

similarly (7.18) and (7.19) imply

$$g^{0} = \mu_1 R^{\phi} \ , \tag{7.29}$$

$$h^{0} = \mu_2 R^{\psi} \ . \tag{7.30}$$

Using (7.12), we may determine σ_1 and σ_2 from (7.21) and (7.22):

$$L^{\phi}\,\mathcal{L}(\sigma_1 R^{\phi}) = 0 \ , \tag{7.31}$$

$$L^{\psi}\,\mathcal{L}(\sigma_2 R^{\psi}) = 0 \ . \tag{7.32}$$

Here the initial values for σ_1 and σ_2 may be prescribed arbitrarily. Once σ_1 and σ_2 are known, we may determine $\mu_1(0;t)$ and $\mu_2(0;t)$ from (7.21) and (7.22):

(7.33) $$L^{\psi}\,[\mathscr{L}a^{0}+\mu_{2}(0;t)R^{\psi}] = 0 \quad ,$$

(7.34) $$L^{\phi}\,[\mathscr{L}b^{0}+\mu_{1}(0;t)R^{\phi}] = 0 \quad .$$

Then a^1 and b^1 may be determined within a multiple of R^{ϕ} and R^{ψ} respectively from (7.16) and (7.17).

Now we can attack the system (7.26). We introduce new variables ξ and η by means of the relations

$$\xi = t - \tau \ ,$$
$$\eta = \tau \quad .$$

Using (7.12), (7.26) becomes

(7.35) $$\begin{cases} \dfrac{\partial}{\partial\xi}\mu_1 + L^{\phi}BR^{\phi}\mu_1 + L^{\phi}BR^{\psi}\mu_2 = 0 \ , \\[2ex] L^{\psi}BR^{\phi}\mu_1 + \dfrac{\partial}{\partial\eta}\mu_2 + L^{\psi}BR^{\psi}\mu_2 = 0 \ . \end{cases}$$

The lines ξ = const. and η = const. are the characteristics of this system. μ_2 is prescribed for $\eta = 0$ by (7.33), and μ_1 is prescribed for $\xi = 0$ by (7.34). Thus μ_1 and μ_2 may be obtained by solving a characteristic initial value problem involving only two independent variables. If we did not assume that a^j, b^j, g^j, and h^j are independent of x and α, then the number of variables involved would be larger.

Once (7.35) has been satisfied, we can determine $g^1(t-\tau;\tau)$ modulo R^{ϕ} from (7.24). Continuing in a

similar fashion, we may obtain a^j, b^j, g^j and h^j in succession if initial data for a^j and b^j are given. For $j > 0$, it will be necessary to solve inhomogeneous ordinary differential equations analogous to (7.31) and (7.32) and inhomogeneous characteristic initial value problems analogous to (7.35), (7.33), (7.34).

The entire expansion for u is determined uniquely as soon as Cauchy data for u are given. The assignment of initial values for a^j and b^j can be made exactly as in Section 3.

8. The Singularity of the Riemann Function on the Hull of the Ray Cone.

In this section, we shall use an expansion of the Riemann function which is valid even for equations with self-intersections of the normal surface. We can apply the method of stationary phase to integrals of the expansions derived in the previous section. The distinctive feature of the Riemann function in the present case is that the singularity of the Riemann function is not confined to the ray cone. Indeed, if a single point of the normal surface is mapped onto two rays, then the Riemann function is, in general, singular on the convex hull of these rays. For strongly hyperbolic equations, the singularity on the hull is weaker by 1/2-derivative than it is at ordinary points of the ray cone.

First we observe that the decomposition of the delta-function given in Section 4, together with the results of the previous section, enables us to represent the singular part of the Riemann function as a superposition of plane waves. In order to obtain the highest singularity of the Riemann function, we may apply the method of stationary phase as described in Sections 5 and 6. The only new feature arises in the treatment of integrals of the form

$$\int_{|\alpha|=1} \int_{0}^{t} f(\phi(t,x;\alpha) + \rho(\alpha)\tau)\, g(t-\tau;\tau)\, d\tau\, d\alpha \,, \tag{8.1}$$

where $\rho(\alpha)$ may vanish on certain curves on the unit sphere.

The condition of stationary phase means that, at the point $\alpha_0, \tau_0 \quad (0 \leq \tau_0 \leq t)$,

$$\phi_\alpha(t,x;\alpha_0) + \tau_0 \rho_\alpha(\alpha_0) = 0 , \tag{8.2}$$

$$\rho(\alpha_0) = 0 . \tag{8.3}$$

Here the subscript α denotes the gradient on the sphere $|\alpha| = 1$. α_0 and τ_0 depend upon t and x. Equation (8.3) shows that, at the point of stationary phase, $\phi(t,x;\alpha_0) = \psi(t,x;\alpha_0)$, i.e. α_0 corresponds to a double point of the normal surface.

In order to interpret τ_0, we first find x_1 and x_2 such that

$$\phi_\alpha(t,x_1,\alpha_0) = 0 , \qquad \phi(t,x_1,\alpha_0) = c ;$$

$$\psi_\alpha(t,x_2,\alpha_0) = 0 , \qquad \psi(t,x_2,\alpha_0) = c .$$

The points x_1 and x_2 are uniquely determined by these equations. If $c = 0$, x_1 and x_2 represent the two rays which correspond to α_0. Since ϕ and ψ are linear in x, the unique solution of the equations

$$\phi(t,x,\alpha_0) = c ,$$

$$\phi_\alpha(t,x,\alpha_0) + \tau_0\rho_\alpha(\alpha_0) = 0 ,$$

is given by

$$x = (1 - \frac{\tau_0}{t})\, x_1 + \frac{\tau_0}{t}\, x_2 \quad .$$

Thus τ_0 is a parameter on the segment joining x_1 and x_2. If x does not lie on this segment, then there will be no point of stationary phase.

For simplicity, we shall treat only the most usual case, where the sheets of the ray cone corresponding to $\phi(t,x;\alpha)$ and $\psi(t,x;\alpha)$ are convex at x_1 and x_2 respectively. Then, if $\xi^1,\ldots,\xi^{n-1}$ are local coordinates on the unit sphere, and $\lambda^1(\alpha_0) = \lambda^2(\alpha_0) > 0$, the matrices

$$\phi_{\xi^i\xi^j}(t,x_1,\alpha(\xi)) \qquad i,j = 1,\ldots,n-1$$

and

$$\psi_{\xi^i\xi^j}(t,x_2,\alpha(\xi)) \qquad i,j = 1,\ldots,n-1$$

are positive definite. Then it follows that if

$$x = (1 - \frac{\tau_0}{t})\, x_1 + \frac{\tau_0}{t}\, x_2 \quad ,$$

then

$$(8.4) \quad \phi_{\xi^i\xi^j}(t,x,\alpha_0(\xi)) + \tau_0\rho_{\xi^i\xi^j}(\alpha_0(\xi)) = (1 - \frac{\tau_0}{t})\, \phi_{\xi^i\xi^j}(t,x_1,\alpha_0) + \frac{\tau_0}{t}\, \psi_{\xi^i\xi^j}(t,x_2,\alpha_0).$$

It follows that the matrix

$$\phi_{\xi^i\xi^j}(t,x;\alpha_0) + \tau_0\rho_{\xi^i\xi^j}(\alpha_0) \qquad i,j = 1,\ldots,n-1$$

is positive definite. We may choose $\xi^1,\ldots,\xi^{n-1}$ such that this matrix is diagonal.

Now we may apply the method of stationary phase to (8.1), integrating first with respect to α. It is easily seen that

$$(8.5)\quad \frac{\partial^2}{\partial\tau^2}[\phi(t,x;\alpha(\xi(\tau)) + \tau\,\rho(\alpha(\xi(\tau)))]$$

$$= -\sum_{i=1}^{n-1}\frac{\rho^2_{\xi^i}}{\phi_{\xi^i\xi^i} + \tau\rho_{\xi^i\xi^i}}\ .$$

Hence, applying Lemma 5.2 and setting $\pm = \operatorname{sgn}\phi_t$,

$$(8.6)\quad R \sim I_+^{\frac{2-n}{2}}\ \delta(\pm\tilde{\phi})\left[\frac{i^{n-2}\ \pi(2\pi)^{n/2}\ g(t-\tau_0;\tau_0)}{\prod_{i=1}^{n-1}\left|\phi_{\xi^i\xi^i}+\tau_0\rho_{\xi^i\xi^i}\right|^{1/2}\left(\sum_{i=1}^{n}\frac{\rho^2_{\xi^i}}{\phi_{\xi^i\xi^i}+\tau_0\rho_{\xi^i\xi^i}}\right)^{1/2}}\right].$$

Here $\tilde{\phi} = \phi(t,x;\alpha_0)$. The same formula holds for both odd and even n. It should be remembered that this term arises only if x lies on the segment joining x_1 and x_2, i.e. only on the hull of the rays corresponding to α_0.

Bibliography

[1] J. Bazer and O. Fleischman, Propagation of weak hydromagnetic discontinuities. Research Rep. No. MH-10, New York University, 1959.

[2] R. Courant and D. Hilbert, Methoden der Mathematischen Physik, II. Berlin, 1937.

[3] R. Courant and P. D. Lax, The propagation of discontinuities in wave motion. Proc. Nat. Acad. Sci. 42 (1956), 872-876.

[4] F. G. Friedlander, Sound pulses in a conducting medium, Proc. Camb. Phil. Soc., 55 (1959), 341-367.

[5] K. O. Friedrichs and H. Kranzer. Notes on Magneto-hydrodynamics VIII. Nonlinear wave motion. Research Rep. No. NYO-6486, New York University, 1958.

[6] H. Grad, Propagation of magnetohydrodynamic waves without radial attenuation. Research Rep. No. NYO-2537, New York University, 1959.

[7] F. John, Plane Waves and Spherical Means. New York, 1955.

[8] J. B. Keller, A geometrical theory of diffraction, Calculus of Variation and its Applications, Proc. of Symposia in Applied Math., vol. VIII, pp. 27-52, New York, 1958.

[9] R. E. Langer, Turning points in linear asymptotic theory, Research Report No. 127, University of Wisconsin, 1960.

[10] A. Lax, On Cauchy's problem for partial differential equations with multiple characteristics. Comm. Pure Appl. Math. 9 (1956), 135-169.

[11] J. Leray, Uniformisation de la solution du problème linéare analytique de Cauchy près de la variété qui porte les données de Cauchy. Bull. Soc. Math. France 85 (1957), 389-429.

[12] J. Leray, La solution unitaire d'un opérateur différentiel linéare. Bull. Soc. Math. France 86 (1958), 75-96.

[13] J. Leray, Le calcul différentiel et intégral sur une variété analytique complexe. Bull. Soc. Math. France 87 (1959), 81-180.

[14] D. Ludwig, Exact and asymptotic solutions of the Cauchy problem. Research Rep. No. NYO-2545, New York University, 1959.

[15] D. Ludwig, Conical refraction in crystal optics and hydromagnetics. Research Rep. No. NYO-9084, 1960.

[16] H. Weitzner, On the Green's function for two-dimensional magnetohydrodynamic waves. Bull. Am. Phys. Soc. Sec. 2, 5 (1960), 321.

[17] M. Yamaguti and K. Kasahara, Sur le système hyperbolique a coefficients constants. Proc. Japan Acad. 35 (1959), 547-550.